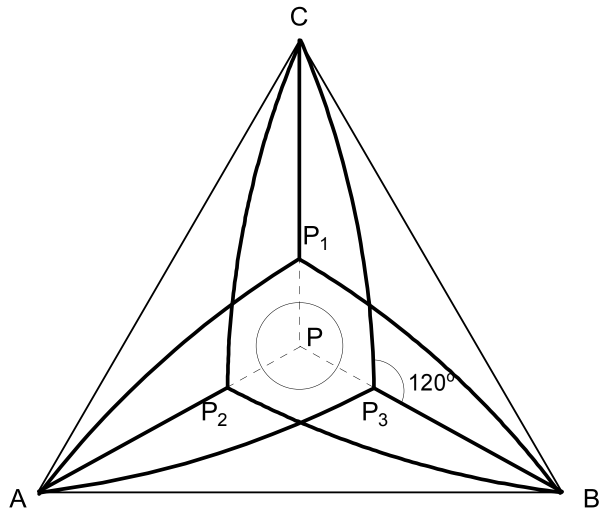
Figure 8-20 Multiple least cost Steiner networks

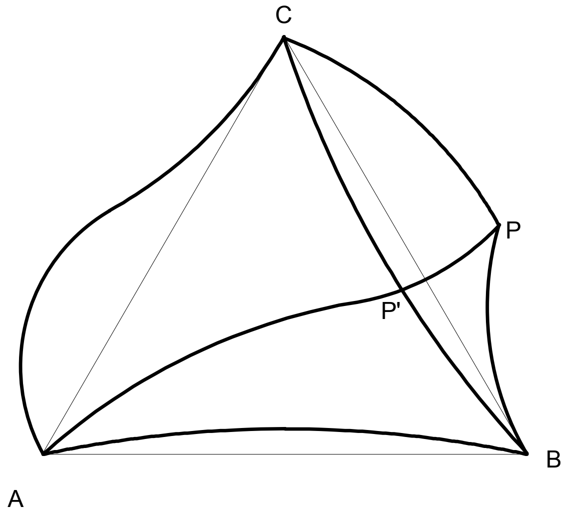


(ii) Let *ABC* be any triangle in the plane with internal angles less than 120°. Then:

(a) a generalised Steiner network may have no Steiner point inside[37] the geodesic triangle *ABC* on the surface S^*; Proof: the geodesic triangle may have vertex angles which are completely different from the plane triangle *ABC*. For example, the vertices could be collinear in S^*.

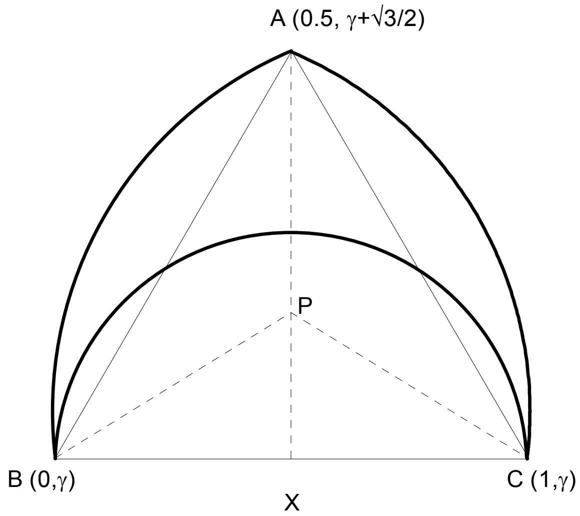
(b) every minimal network will be comprised of arcs lying inside or on the geodesic triangle *ABC* (where geodesic paths are defined as the globally shortest paths connecting the vertices). Proof: (see Figure 8-21) if *P* is a Steiner point lying outside the geodesic triangle then the path from *P* to at least one of the vertices must cross one side of the triangle at *P'*, say. Now length *AP'B*<*APB* thus a network with *P'* as a node is a lower cost solution than the network with *P* as a node.

Figure 8-21 Steiner points in a geodesic triangle



We may consider application of (ii) to the analytically tractable problem in which $F=\alpha/y$ where $\alpha>0$. Solving the differential equation at the start of this subsection (see Section 12.1.2.2 for more details) we find that least cost paths are arcs of circles (or straight lines in the extreme). Now consider the equilateral triangle (Figure 8-22) across which this cost function $F=\alpha/y$ has been defined. By symmetry we see that if a Steiner point is to be found it must lie along the line *AX*. Furthermore, network arcs must intersect at 120°. When $\gamma=0$ we find that angle *BAC* of the geodesic triangle equals 120° and the unique minimal network is the minimal spanning tree. As γ increases a Steiner point, *P'*, will be found on *AX* between *A* and *P*. For sufficiently large γ *P'* will coincide with *P* and the curved edges of the geodesic triangle will become straight lines.

Figure 8-22 Steiner network Q_3 in the field $F=\alpha/y$



Taking points (i) and (ii) above into account an approximate solution procedure for the generalised Q_3 problem may be defined (Figure 8-23):

(i) find the geodesic paths which together make up the geodesic triangle *ABC* using boundary value, initial value or distance transform methods

(ii) divide two of the three vertex angles into $n+1$ (n variable) subdivisions defining n new directions at each vertex selected

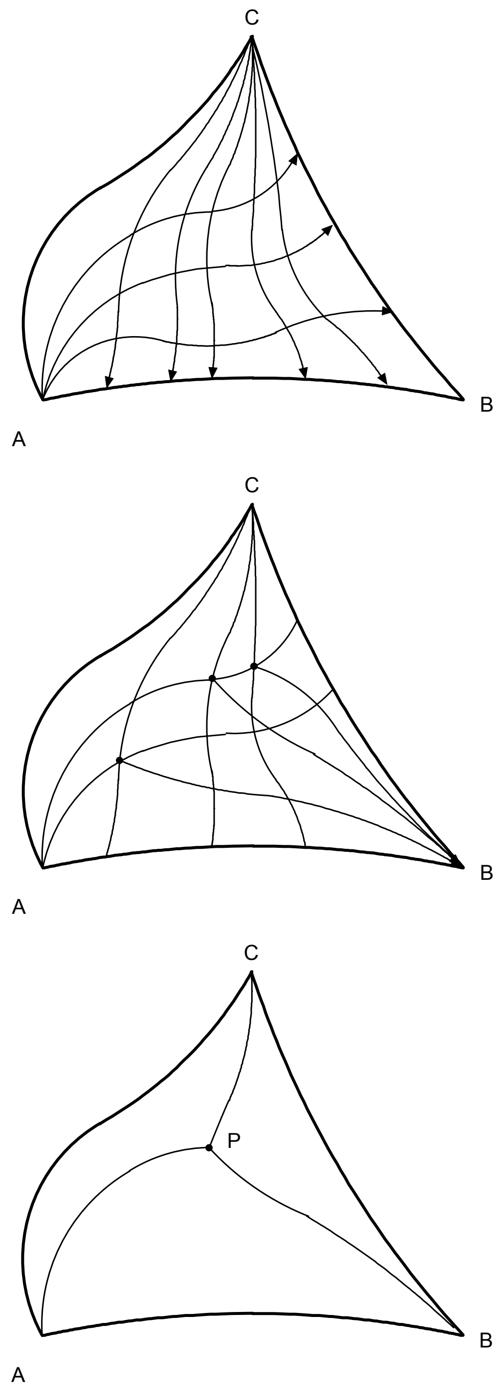
(iii) for the two selected vertices, generate the $2n$ geodesic paths which are defined by the selected vertices and the initial directions specified in (ii), which lie within the geodesic triangle *ABC*

(iv) from the $N \leq n^2$ intersections of solution paths, select the subset of M points for which the intersections occur at approximately 120° (additional interpolation may be required to achieve a satisfactory approximation).

(v) solve the M boundary value problems defined by these intersections and the final vertex, or the initial value problems defined by the M intersection points and a direction or 120° from both the lines which have created the intersection.

(vi) if the initial value problems in (v) give a solution (one or more) that intersect the final vertex then the solution has been found. Alternatively, select the boundary value solution(s) that give the initial angle result as close as possible to 120°.

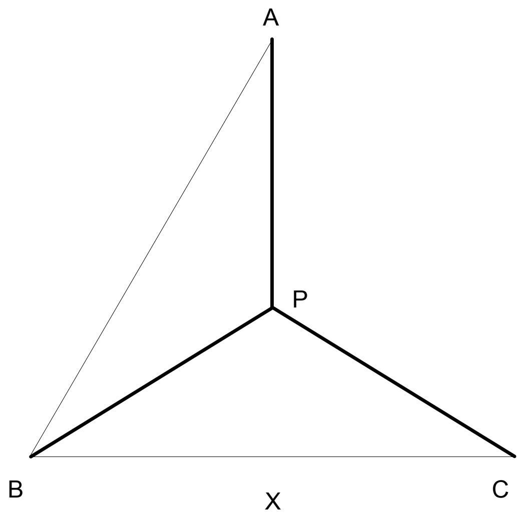
Figure 8-23 Solution procedure for generalised Steiner networks



The method described can also be applied to cases where the weights, w_j, are variable, by solving for the Steiner point in the plane, thereby determining the angles at the Steiner point, and using these for the intersections. In practice, however, the complex form of physical surfaces and generalised cost surfaces in the real world coupled with multiple constraints, means that alternative procedures such as the Vortal technique described earlier will be the preferred approach. In this case, the geodesic characteristics of tension and straightness (parallel transport) are more useful tools than localised angular behaviour.

As solutions to Q_3 are not generally unique, solutions to Q_n cannot be found by examining all possible topologies. An heuristic approach may, however, be suggested. First, calculate T_n, the generalised *mst*; second, select sets of vertices from T_n and find generalised Steiner points for these subsets using the procedure described above for three nodes (where nodes are vertices or Steiner points). Finally use the newly selected Steiner minimal sub-networks to replace the original sections of *mst*, reducing the overall length of the *mst*. Continuation of this selection and replacement procedure will, in general, produce a sub-optimal but improved solution network.

Figure 8-24 Steiner minimal network in the plane, Q_3



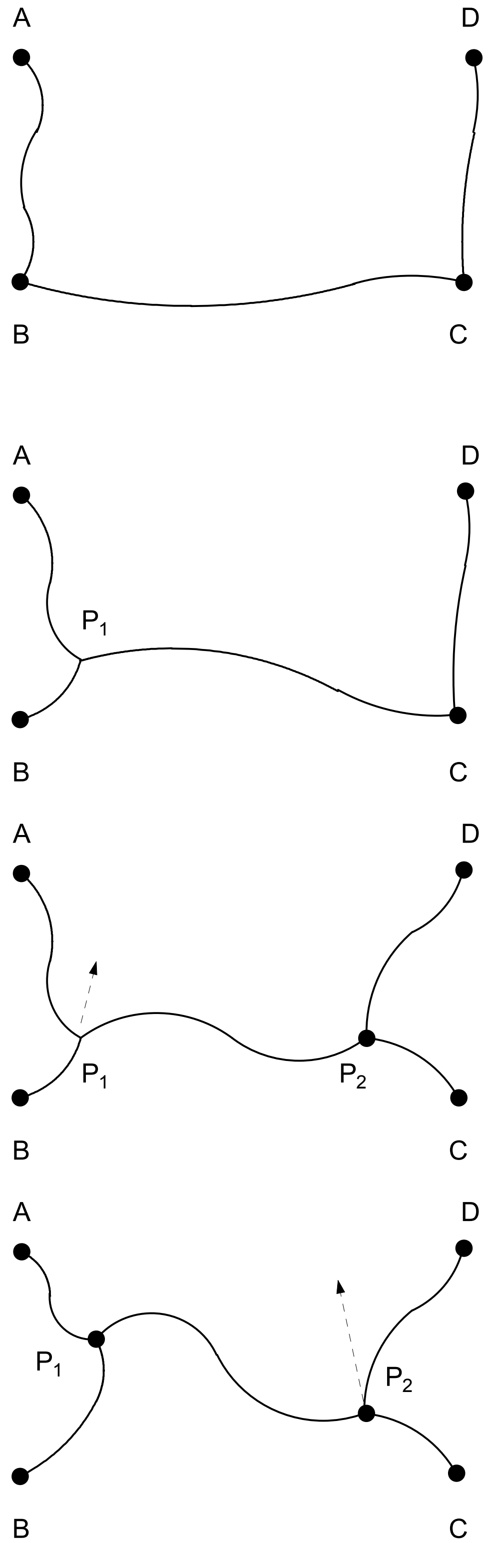
To illustrate this process consider the cases *n=3* and *n=4*.

For *n=3* T_3 is first solved by providing a ‘reference’ solution network with two arcs connecting three vertices (Figure 8-24). A Steiner minimum network Q_3 is then found, using the methods described above, and the length (cost) of this new network is compared with that of the *mst*. The shorter (lower cost) is chosen as the solution network. In the diagram above (constant costs in the plane), the path lengths $AB+BC >$ $AP+ BP+ CP$ so the Steiner network solution is selected with point P as a new intermediate node, at a saving of just under 15%.

For n=4 the *generalised mst* T_4 will have three arcs (Figure 8-25a) connecting the four vertices $\{A,B,C,D\}$. We then select a subset of three vertices, $\{A,B,C\}$ say, and solve the Q_3 problem for these three using the methods described above (Figure 8-25b). If the new network, ABP_1CD is shorter than the length of T_4 we have an improved, though generally sub-optimal solution to the Q_4 problem. Treating the point P_1 as fixed we now solve the Q_3 problem for P_1CD yielding a new network with two Steiner points, P_1 and P_2 (Figure 8-25c). Note that P_1 will generally no longer be optimally located with respect to the 'fixed points' A, B and P_2 and may be re-adjusted using these three points. Iterative adjustment of P_1 and P_2 continues until there is no worthwhile reduction in the length of the network and this is then taken as the solution for Q_4.

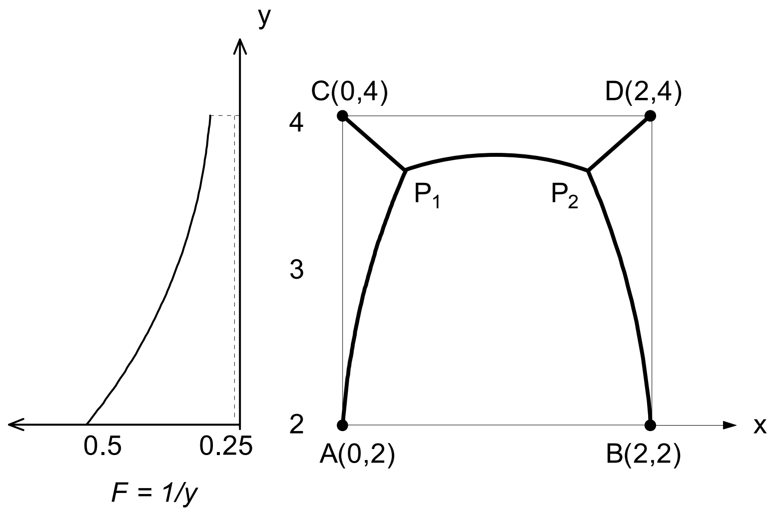
Had our original choice of subset been $\{A,C,D\}$, $\{B,C,D\}$ or $\{A,B,D\}$ we might have reached different solutions to Q_4, possibly shorter (more optimal) than the result achieved using $\{A,B,C\}$. Similarly, solving the intermediate problem for $\{P_1,A,D\}$ and dropping the link CD could have yielded another network topology. The number of possible approaches to tackling Q_4 is large, but for reasonably behaved cost functions the different approaches should yield similar solutions in terms of total cost. For larger problems T_n may be found, separated into smaller sub-networks, and the approaches described for Q_3 and Q_4 applied to these, as before. In essence, however, the solution process will remain one of trial and error, exploring alternative topologies and investigating the sensitivity of networks to alterations in the cost function and weights.

Figure 8-25 Solution procedure for the generalised Q_4 problem



The solution to Q_4 when the cost function $F = 1/y$ (cf. $F=1/v$ cited earlier, where v is velocity) can be found directly since we know that geodesics will be arcs of circles. For the example shown (Figure 8-26) a unique topology with two Steiner points exists, with a total length approximately 20% lower in cost than the *mst* and 10% less than the Steiner solution in the plane.

Figure 8-26 Solution of the Q_4 problem with $F = 1/y$



The above discussion may also be applied to the problem in which vertices are weighted. In this case the angles at which arcs meet will no longer be 120°. The solution procedure for Q_3 may be modified as follows:

(i) solve for the incident angles of arcs at the Steiner point in the plane with constant costs (i.e. using the resultant of the weighted vectors)

(ii) use these angles to select the M boundary points and final solutions, as in the previous method.

For example, if the three vertices are weighted 1, 1 and 0.5 then the angles between the arcs will be 151°, 104.5° and 104.5°. Selecting the vertices with unit weight and solving the initial value problems, a set of M 151° intersections will be found. M boundary value problems are then solved using the selected intersections and the remaining vertex, with solutions yielding 104.5° inside the geodesic triangle being selected as generalised Steiner points.

Further extensions along these lines are possible. For example: application of the methods developed to problems of rationalising existing networks; generalisation of the L_p metric approach allowing p to be (piecewise) continuous function of location $p =$

$p(x,y)$; and introducing solution procedures based on computational algorithms such as Vortal, VTDT and LCDT.

Analysis of fundamental problems in network design and location theory has shown that analytic and computational solution methods can be successfully applied when the assumption of isotropy is dropped. Distances and paths can be calculated in a variety of ways that produce optimal or near-optimal solutions. Much work remains to be undertaken in this field but it is now evident that complex, realistic problems can be undertaken using tools that hitherto have not been available.

Notes and References:

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[1] *parts of the text of this Chapter were published by the present author in 1981, as:*
de Smith M J *(**1981**) Optimum location theory – generalisations of some network problems and some heuristic solutions, J. Reg. Sci., 21, 4, 491-505*

[2] *due to Fermat and Gauss but studied in detail by Steiner and Weber, and also known as a Minimum Aggregate Transport or MAT problem. The literature on network design, optimisation and approximation problems is extremely large, partly because it has many applications and partly because it is eminently suited to analysis. Approximation methods have a very recent history, only dating from Johnson's paper in 1974. Johnson is Head of the "Algorithms and Optimization Department", AT&T Research. See for example:*
Johnson D S *(**1974**) Approximation algorithms for combinatorial problems, Journal of Computer and System Sciences, 9, 256-278*
In the last few years a Compendium of results (with almost 500 references) has been prepared and maintained on an Internet web site:
http://www.nada.kth.se/~viggo/wwwcompendium/wwwcompendium.html.

[3] ***Miehle W*** *(**1958**) Link length minimisation in networks, Operations Research, 6, 232-243*
Scott A J *(**1971**) Combinatorial programming, spatial analysis, and planning, Methuen, London*
This problem was originally proposed as a 'challenge' to other mathematicians by Fermat in the 17th century

[4] ***Steiner points**: Such points are on occasion referred to as **Weber points** or point of minimum aggregate travel (**MAT**) but these latter terms apply to the more general problems associated with location theory in which there are any number of vertices and variable weights and/or transport costs in each case. Steiner network problems are sometimes referred to as 'floating node' problems, in contrast to the fixed node problems in which the location of all vertices are pre-defined.*

[5] *see for example: **Hamacher H W, Klamroth K** (**2000**) Planar Weber location problems with barriers and block norms, Annals of Operations Research, 96,191-208*

[6] *see, for example: **Warme D M, Winter P, Zachariasen M** (**1998**) Exact algorithms for planar Steiner tree problems: A computational study, Tech. Rpt. DIKU-TR-98/11, Dept. of Computer Sci., University of Copenhagen, Copenhagen*

[7] ***Love R F, Morris J G, Wesolowsky G O*** *(**1988**) Facilities location, models and methods, Ch. 2.3, North-Holland, New York*

[8] *for a more recent and extensive discussion of solutions for general values of p in polynomial time, see **Xue G, Ye Y** (**1997**) An efficient algorithm for minimizing a sum of p-norms. SIAM Journal on Optimization, 10, 551-579*

[9] *lines of equal cost are known as **isodapanes** - a cost surface may be represented graphically by a set of isodapanes, which in the Steiner case will form closed elliptical curves around a single minimum point.*

[10] ***Gilbert E N, Pollock H O*** *(**1968**) Steiner minimal trees, J. Soc. Indust. Applied Maths.,14, 255-265*

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[11] ***NP:*** *This terminology derives from that part of computer science and mathematics concerned with computability and complexity - in broad terms problems that are NP-hard or NP-complete are either extremely difficult or impossible to solve using any efficient (fast, small) algorithm. The Steiner tree problem with Euclidean metric is known to be NP-hard and a second common network optimisation problem, the Travelling Salesman Problem (TSP), is known to be NP-complete.*

[12] ***Massam B H (1980)*** *Spatial search: Application to planning problems in the public sector, Pergammon, Oxford, p.xix*

[13] ***Hwang F K (1976)*** *On Steiner minimal trees with rectilinear distance, SIAM J. of Applied Math., 30, 1, 104-114*

[14] ***Drezner Z, Wesolowsky G O (1989)*** *The asymmetric distance location problem, Transportation Science, 23, 201-207*

[15] ***Bose P, Maheshwari A, Morin P (2002)*** *Fast approximations for sums of distances, clustering and the Fermat-Weber problem, Pending: accepted for publication in Computational Geometry: Theory and Applications*

[16] ***LOLA home page:*** *http://www.mathematik.uni-kl.de/~lola/ (Kaiserslautern University). This software suite includes solution procedures for: planar, network (directed and non-directed) and discrete problems; with 1 to n facility locations; a range of metrics which include l_1, l_2, l_p, l_∞ and gauge (block norm) distances for planar problems and vertex or on-graph distances for graphs; with a range of objective functions (media, max etc). See LOLA Manual, Chapter 3, for a summary of algorithms provided*

[17] ***Ratti C (2001)*** *Urban analysis for environmental prediction, Unpub. PhD Thesis, University of Cambridge Dept. of Architecture, Cambridge, England*

[18] ***Batty M, DeSyllas J, Duxbury E (2002)*** *The discrete dynamics of small-scale events: Agent-based models of mobility in carnivals and street parades, Working Paper 56, Centre for Advanced Spatial Analysis, Univ. Coll. London*

[19] *statistical analysis of DT matrices first excludes all those elements (cells) which are obstacles, have been found to be inaccessible or are part of the target set*

[20] *a similar procedure has been described by* ***Kimmel R, Kiryati N, Bruckstein A F (1994)*** *Sub-pixel distance maps and weighted distance transforms, J. Math. Imaging and Vision, Special Issue on Topology and Geometry in Computer Vision*

[21] *DTs are ideally suited to parallel processing and in some instances, firmware implementation. Multi-level DTs of the type described here are clear candidates for such treatment. Very fast exact algorithms are now also available for sequential processing and examples are provided in Annex 3*

[22] *examples using EDTs provided within the MATLAB image processing toolbox are also provided at http://ww.desmith.com/MJdS - in these examples the display is in image coordinate order rather than geographic coordinate order*

[23] *time slices of cost fields could be derived in many ways, for example: from prior collected datasets, from dynamically gathered data, or determined by equations of evolution applied to the two-dimensional field (e.g. a diurnal model of traffic congestion).*

[24] ***Gould P R (1966)*** *Space-searching procedures in geography and the social sciences, Univ. of Hawaii Social Science Research Institute, Paper 1, Hawaii*

Manheim M L (1964) *Highway route location as a hierarchically-structured sequential decision process; an experiment in Bayesian theory for guiding an engineering process, MIT Civil Eng. Lab., Res. Rpt R64-15*

Rowe N C, Ross R S (1990) *Optimal grid-free path planning across arbitrarily contoured terrain with anisotropic friction and gravity effects, IEEE Transactions on Robotics and Automation, 6, 540-553*

[25] ***Kuffner J J, LaValle S M (2000)*** *RRT Connect – An efficient approach to single query path planning, Proc. IEEE Int'l Conf. on Robotics and Automation (ICRA'2000), San Francisco, CA, April 2000*

[26] ***RRT-connect algorithms****: are probabilistically complete (i.e. if a solution exists they will find it with probability 1 given sufficient iterations) and using simulations rapid convergence (improving on most/all other current algorithms for comparable problems) has been demonstrated but not proved by Kuffner and LaValle (op. cit.).*

Barraquand J, Kavraki L. et al (1996) *A Random Sampling Scheme for Path Planning, 7th Intern. Symp. on Robotics research. Barraquand et al show that with the less-efficient uniform random sampling methods for probabilistic roadmap construction the number of samples, s (vertex points), required to guarantee a solution path is found if one exists is proportional to $1/R^2 ln(L/R)$, where R equates to the minimum step-length used in the RRT algorithms discussed and L is the solution path length. For a 1km path with 10metre step length, the number of samples, s < 5000.*

[27] ***Road gradients****: guidelines vary as to the maximum acceptable gradients for different kinds of road and paths (for pedestrians, cyclists and horses). For UK Motorways, such as the M6, the constraints applied were: Curves, Radius not less than 3000ft; Gradients, Normal maximum 1 in 30, but up to 1 in 20 to be permitted in some hilly country. Very few roads currently constructed in the UK will have a gradient steeper than 1:10 and 1:15 or less is the norm. Vertical curvature limits are also normally specified.*

[28] ***Hanan M (1966)*** *On Steiner's problem with rectilinear distance, J. Soc. Indust. Applied Maths., 14, 1-29*

[29] *Useful additional references not included in Johnson's bibliography cited above include:*

Gilbert E N, Pollock H O (1968) *Steiner minimal trees, J. Soc. Indust. Applied Maths.,14, 255-265*

Melzak Z A (1964) *On the problem of Steiner, Canad. Math. Bull., 4, 143-148*

Miehle W (1958) *Link length minimisation in networks, Operations Res., 6, 232-243*

[30] ***Hamacher H W, Klamroth K (2000)*** *Planar Weber location problems with barriers and block norms, Annals of Operations Research, 96,191-208*

[31] ***Huriot J M , Perreur J (1973)*** *Modèles des localisations et distance rectilinéaire, Revue d'Economie Politique, 60, 77-86*

[32] ***Peeters D, Thomas I*** (***1997***) *Distance-L_p et localisations optimales. Simulations sur un semi aléatoire de points, Cahiers scientifiques du transport, 31, 55-70*

[33] ***Eisenhart L P*** (***1925***) *Riemannian geometry, Princeton Univ. Press, Princeton, New Jersey*

[34] ***Gilbert E N, Pollock H O*** (***1968***) *Steiner minimal trees, J. Soc. Ind. Applied Maths., 14, 255-265*

[35] ***Weddell R E*** (***1971***) *Some aspects of the theory of location, Unpub. PhD thesis, Northwestern Univ., Illin.*

[36] ***Katz I N, Cooper L*** (***1980***) *Optimal location on a sphere, Computers and maths with applications, 6, 175-196*

[37] ***Inside****: a point P is considered to lie inside the triangle ABC if P lies to the left of an observer traversing the boundary of ABC in an anticlockwise direction*

9 Conclusions

9.1 Overview

As we have shown in the initial part of this study, from a need for better information and maps of nations, coastlines, continents, the Earth and the heavens, came a drive for new standards, precision in data collection, and the development of new instruments and procedures. Together these developments led to the solution of many of the key practical and theoretical problems in distance and time measurement, and contributed much to our understanding of errors and uncertainty. From these original secure foundations, the current body of knowledge, theories, procedures and systems has been developed, culminating in today's range of tools, GIS technologies and advanced position determination and measurement systems.

However, the majority of the theoretical development has dealt with homogeneous or (radially) symmetric problems. Our study has sought to illustrate and investigate the spatial building blocks of such developments, and then to extend the set of measures and procedures available in order to address problems experienced in practical, real-world (inhomogeneous) situations. Of course, many such problems remain unsolved and others will be unsolvable (at least, within a formal mathematical or algorithmic framework), but a number of promising lines of attack have been developed and hopefully these will start to address the issues expressed by Tobler and Harvey in our opening quotations.

9.2 Key observations

In the preceding Chapters we have explored many aspects of distance and path measurement and this process has enabled us to design new solution procedures for a significant range of problems within this field. However, before considering the outstanding issues and future developments associated with these new procedures, it is useful to highlight some of the key observations raised during the course of our investigation:

Historical foundations: the historical research provides a perspective to the entire study whilst being very different in content from most of the subsequent work. It sought to show how an understanding of measurement techniques (involving both length and time), mathematical formulations, and the development of precision tools (instruments) enabled rapid progress to be made. This in turn facilitated safe navigation and thereby trade, and provided critical foundations for the industrial and information ages.

The research also demonstrated that an obsession with error and accuracy, in the broadest sense, was an essential part of this scientific progress, and is as important today as it was in the 18th and 19th centuries. Echoing the requirements of the original Longitude Act of 1714, we seek theories and methods that are “Practicable and Useful”, i.e. reliable, accurate, practical and repeatable, and, of course, cost effective. It also became clear that substantive progress depended on many interlinked factors, and frequently it was commercial or military pressures rather than pure scientific research that drove this – if anything this is even more the case today than it was in the 18th century.

Metrics: the historical analysis included a discussion on the development of more advanced distance formulae. This led us to examine the range of expressions that may reasonably be used for calculating distance, or more generally the degree of separation of pairs or sets of objects belonging to a set. We demonstrated that there are many valid generalisations of our common notion of distance and a wide range of methods for the computation of distance from coordinate data. The variety of measures used, and their differing attributes, is surprising. A number of new results were presented, including variants and developments of the standard Euclidean and Riemannian formulae.

The analysis led us to conclude that incremental computation of distance measures is a more robust and meaningful approach for many practical problems, despite the considerable increase in effort that may be required to achieve this, and the issues of path selection it raises. The second conclusion in this area is that whatever measures one uses, they must satisfy what we might call the ‘Longitude requirements’ noted above.

Paths: If incremental measures are the preferred forms of distance computation, then attention becomes focused on the notion of path. In our analysis we distinguish between the use of various models to represent real-world linear structures, and the computation of path lengths from such representations. When determining distances we make implicit or explicit assumptions about the path along which measurement is to take place - ratio scale distance measures are generated as a result of a process of accumulating a series of two or more ordered point-pair measurements, and when there are more than two such ordered pairs we have a path.

There are many possible and reasonable representations or models of path, some of which do not support the notion of distance (length) at all. Whilst this latter observation has a clear and distinct importance, both from a practical and theoretical perspective, we conclude that it does not provide a fundamental block to development of distance-related analysis, any more than an appreciation of uncertainty in particle physics or the complexities of the integers in number theory acts as a block to research and development in these fields. In many ways, such issues have the reverse effect, stimulating analysis and exploration of the issues, leading to new insights, theories and procedures.

Statistics: Distances are determined from point-pairs, but the location of these points may not be known precisely. This leads to two linked sets of research and results: the first is to compute the expected or average separation of point-pairs under a variety of conditions and metrics; and the second is to determine from a given set of measurements the nature of the underlying distribution of points. This process is technically complex, and whilst a substantial variety of results are presented and derived, the prospects for their further development using analytic models appears limited. The use of simulation methods, using analytic results as controlled test models, would appear to be one of the most productive routes forwards. Such methods must have their underlying assumptions, generation procedures and parameters laid out in detail, much as with analytic models, but there is little reason to reject such approaches as somehow ‘less acceptable’ than analytic results. There is also much merit in their use as a set of experimental tools from which new datasets, tabulated results, theories, test procedures and models may be developed. Indeed, Monte Carlo simulation may well be the best approach to obtain meaningful tests of randomness or of alternative process hypotheses. As noted earlier, the availability of such facilities within GIS would greatly simplify modelling and analysis, since such software typically holds the details of point events, transport networks and the details of boundary form.

Surfaces: There has been very little research into distance computations across generalised surfaces – whether physical surfaces, cost surfaces or combinations of the two – this is a broad area warranting a considerable amount of further analysis and methodological development. Such research could lead to the development of a range of dynamic models and visualisation tools, both areas of great value that would benefit from new theories, techniques and toolsets. Methods based on optical analogies were shown to have a number of drawbacks and alternative procedures based on accumulated cost surfaces (ACS) were found to be preferable. ACS methods were also shown to be subject to error in some instances and of applicability to a restricted range of problems. However, development of the ACS notion in the form of Distance Transforms (DTs) and extension to DT methods was shown to be a promising step forwards.

A second family of procedures was also shown to provide an effective approach to problems that are not suited to DT methods. These are based on a combination of search-and-optimise procedures. One such approach, based on random tree generation and variational optimisation was developed in some detail. We discuss outstanding issues relating to both of these developments in the final parts of this concluding Chapter.

Networks and optimal location: In Chapter 8 we applied a number of the metrics, path models, and surface analysis results and techniques we devised in earlier Sections to standard problems in spatial analysis. The results obtained and extensions to procedures that we describe, serve to confirm the power of approaching such problems by careful analysis of their building blocks. There is enormous scope to develop and test such ideas further – for example: to develop approximation models which provide results that are both fast and asymptotically exact; to construct realistic and usable spatial choice models building upon the MWDT procedure; to construct dynamic traffic modelling and forecasting models utilising the LCDT approach; and to construct and evaluate models of networks under conditions of varying spatial cost. To this extent, the work in the present study has the potential to be very much the start of a development programme.

9.3 Distance transforms

The results and examples provided in the preceding Sections have shown that the various Distance Transform (DT) procedures that we have developed provide a simple, powerful and extensible set of tools for spatial analysts and decision-makers. These procedures facilitate the handling of a wide variety of incremental Euclidean distance problems, including a range of image processing and visualisation applications, together with applications such as buffering and the determination of Voronoi regions (standard DT methods). Extensions to the basic DT algorithm support solution of least cost/time problems (LCDT methods), optionally including spatial constraints such as obstacles and no-go regions, and weighted multi-criteria (MWDT) problems. We have also shown that related DT methods can be used to determine geodesics on physically variable surfaces (VTDT). These various methods may be applied separately or in combination, and are ideally suited to implementation within current GIS software packages.

A central and open question that remains is the stability and convergence of such procedures – experimental evidence suggests that DT procedures for inhomogeneous space problems are intrinsically convergent, but the generality of this finding and the quality (global nature) of such convergence requires close examination. Likewise, the dependence of computational methods on input parameters, the models used for representation of underlying data, dynamics, and scale are all areas for which much research is warranted.

In applying standard and modified DTs of the kinds described in this study the following issues must be considered:

(i) if a standard form of DT is used (i.e. there are no variations in cost, topography nor any obstacles) a predetermined number of passes of the masks will be required (one forward and one backward pass in the case of the simplest DTs)

(ii) if the conditions in (i) do not apply, the solution procedure requires iteration to ensure convergence of the solution values. Experimental evidence suggests that for most problems (potentially all unconstrained variable cost models) two iterations suffice (i.e. one complete extra set of scans; this can be readily demonstrated, for example, using the problem illustrated in Figure 7-5). However, for some, more complex, problems up to 6 (or occasionally more) iterations may be required. Computations based on the VTDT algorithm indicate that a larger number of iterations may be required for this procedure. No theoretical analysis of convergence has been carried out at this time and this remains as an area for further research

(iii) solutions will not converge to regions that are unreachable, i.e. to regions which are completely hidden behind impermeable barriers, very high cost zones and/or on sections of street networks not connected to the main area of study

(iv) if the size of the sample lattice (or sub-sections of this lattice that are surrounded by barriers) is not greater than the DT mask size, then some or all distance values may not be updated and thus the procedure may be invalid

(v) having generated the DT for a specific problem path computation may be required – least cost or time paths will be orthogonal to the optimal DT isolines, and these will not necessarily exactly match isolines generated from an exact DT algorithm. Furthermore, isolines and paths will be distorted by the lattice representation and may not follow local lines that are orthogonal to the isolines. If necessary path determination may be performed using smoothed isolines and/or a retained record of the optimal path, e.g. by storing additional data as part of the DT algorithm (as illustrated in Figure 5-20B)

Despite these concerns, we believe that the various DT procedures we have developed provide one of the most powerful computational approaches to problems in spatial analysis yet devised. Development of toolsets based on simple or exact DTs for use in conjunction with GIS packages is one area for early consideration. Another is the examination of further extensions to the family of models we have developed together with a more detailed exploration of their convergence and precision. It is also anticipated that extension of these procedures to cover gradient and curvature constraints will be a fruitful area for further research.

9.4 Search procedures

DT procedures can be seen as a form of sweep search, exhaustively computing results using a lattice approximation to the underlying spatial data. It may be possible to develop DT techniques to handle more complex path-related constraints, but search procedures based on path growth seem more likely to yield effective solutions in these cases. In this study we have developed one such procedure, VORTAL, based on random tree constructs, which are then iteratively optimised. We have shown how such procedures can solve simple planar path problems and multiple inter-linked path problems, with or without obstacles, in an optimal manner. We have also shown that simple constraints, such as proximity and gradient constraints, can be incorporated into such models and yield high quality solutions, assuming that solutions are forthcoming.

Once one introduces multiple constraints the question of solvability becomes of increasing importance. Potentially any number of paths can be constructed across a continuous unbounded region (or surface), of variable topography and/or cost, of finite length (i.e. ignoring fractal considerations). One or more of these paths *may* be of shortest length. However, there is no guarantee that such a path will be a smooth curve or will avoid regions of steep slopes. When such constraints are added the solution path may be required to take a far longer path, and in many instances the combination of curvature and gradient constraints may mean that there is no solution path possible. In such cases either one or more of the constraints must be relaxed, the project abandoned, or the implicit constraint that the path must be embedded in the surface must be amended. In the latter case tunnels, bridges or cut-and-fill constructs are required. The inclusion of such facilities within the VORTAL procedure would appear to be feasible but would require substantial modification of the algorithm and code developed to date. We see our development of the VORTAL procedure as very much the early stages of a development programme, with the next stage being application to a selection of real-world problems. The latter will result in many enhancements to the procedure, both in terms of functionality and performance – finding better ways of selecting good potential routes and improved methods of optimising the alternatives and evaluating topological variants.

9.5 Overall conclusions

In spite of the progress made, there remain many questions that require closer attention. Amongst these are: the treatment of vector fields and dynamic data; the thorough analysis of solvability and convergence; the formal analysis of selected problems involving constrained optimisation; the applications of the methods developed within spatial analysis and GIS and within other disciplines; and the changing importance of distance in current and future human (terrestrial) activity. In the Postscript, which follows these Conclusions, we comment briefly on one aspect of this latter issue, distance and telecommunications. In connection with this area we argue that telecommunications has a special role, similar to that of the many types and speeds of transport systems, in distorting the isotropic and simple metric view of the world, especially with respect to information-centric activity.

If one were to seek a single, overall conclusion from the research programme undertaken in this study, it would be that the notion of distance, and the application of distance measures to practical problems, is far more complex than it appears at first sight! However, the process of analysis we have undertaken, which examines and then builds upon fundamental spatial components such as location, path and distance, we have found to be both stimulating and we hope, productive.

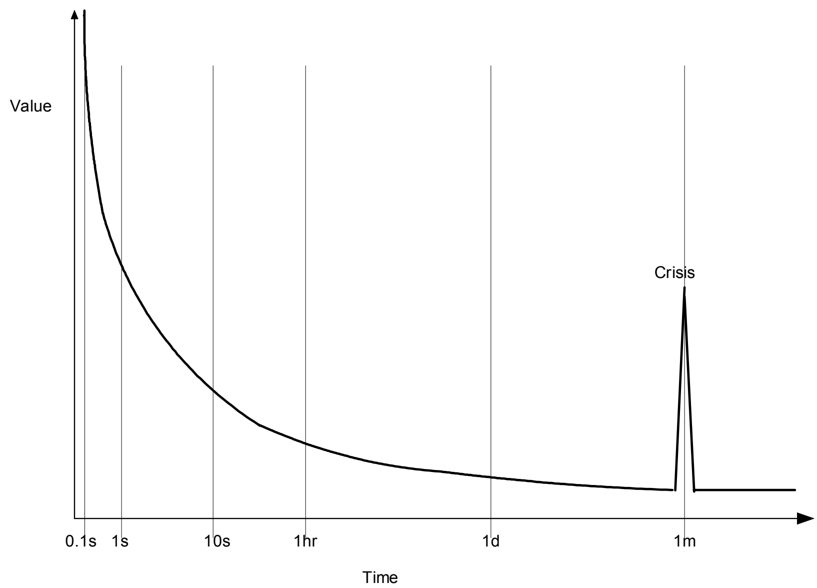
10 Postscript: Distance, telecommunications and timeliness

Continuous technological development has led to the position whereby the speed of information transmission has resulted in the virtual elimination of geographic (spatial) considerations for some applications. Its earliest effects were felt with the development of the telegraph and radio communications, with application to longitude determination as noted at this start of this study. But with timely information being at the centre of military, commercial and even personal decision-making, electronic communications technologies have become central to a large number of spatial processes.

Historically, distance has been a major factor in determining the speed of information delivery, thus influencing location decisions in a complex and dynamic (iterative) manner. For example, early in the 19^{th} century, pigeons were very successfully used as express carriers of news – Rothschild’s Bank in London who provided the financial backing for Wellington’s forces in 1815 were the first in London to be informed of the outcome of the Battle of Waterloo as a result of having their own private pigeon post service from the front line. Thirty five years later Julius Reuter developed a niche business delivering the latest news and information - he used 45 pigeons to carry news and stock prices between Brussels in Belgium and Aachen in Germany within 2 hours, beating the railroad by six hours - the fastest route by road today is approximately 90 miles in length and takes just under 2 hours to drive, excluding city centre congestion. Reuter’s venture was, in reality, a temporary ‘bridging’ operation, filling in the gap in the new telegraph lines that had been laid between Paris and Brussels, and between Berlin and Aachen. Reuter’s success led him to move to London in 1851, then the financial centre of Europe, where he established the news agency that subsequently became the leading financial news and price distribution service in the world. There was cause and effect here – the opportunities resultant upon a communications ‘innovation’ led Reuter to move to London, but once in London Reuter (and others) provided the information and communications systems that enabled the City of London to dominate the world's financial, currency and commodity markets for over a century. London became a global communications node, and remains so in many market areas.

The value, and frequently the cost, of information are dependent on many factors, but assuming that the content meets the requirements of users, then timeliness and accessibility are major factors to be considered. Very valuable information is frequently either information that has little value after a short while ('yesterday's newspaper') or information that is old but required at very short notice. An example of the latter is in handling emergencies – political, economic, medical etc. – which require very fast response and access to a large variety of information and transaction systems at short notice. In a similar (but more dramatic fashion) to distance decay functions, there is an equivalent (and often inter-related) information value time decay function from seconds through hours, days and months, of the form shown in Figure 10-1:

Figure 10-1 Time decay of information value



Many financial information services focus on the far left-hand side of the diagram. Reuters and others now deliver such data across global private networks using proprietary network protocols in order to sustain real-time throughput levels, giving sub-second data delivery simultaneously across their customer base. The data centres are linked by dedicated international communications circuits, which rely on satellite links, optical fibre cables and coaxial cables. Communication between data centres and subscribers is usually by dedicated terrestrial circuits that are leased from telecommunications operators and are supplemented by a variety of other transmission systems, such as satellite-based networks. In practice, this high investment in infrastructure means that whilst service can be delivered almost anywhere in the world, including via the Internet, both the information provision infrastructures and content are expensive. Furthermore, it means that delivery at given guaranteed level of service is far more straightforward in the major financial centres than in provincial towns or less developed countries – for example, data delivery via geostationary satellite facilities is less reliable, serves northerly latitudes poorly, has inherent transmission delays, and in most instances does not provide interactive communications facilities, thus requiring augmentation by terrestrial links (e.g. for transaction handling). This places many regions of the world at a severe disadvantage in terms of such services.

There are many other factors which result in concentration of facilities and services in major centres, despite telecommunications advances – high amongst these are the need for systems integration, security considerations and the necessity for face-to-face contact within and between businesses working in the same or similar markets. Whilst such factors are reducing in importance, and for example open outcry trading floors are closing down, it will be a long time before they disappear completely.

The demise of many global competitors to Reuters is to some extent a testament to the company’s vision and technological innovation, but also reflects two key trends in this most demanding of marketplaces – globalisation and technological change. The former requires suppliers to deliver services everywhere in the world, with global content coverage, in a consistent and guaranteed manner. The latter exerts pressure upon all suppliers by reducing the unit cost of delivery for a given amount of information. One of the side effects of this is the growth of niche market players (rather as Reuter himself started in the 19th century) whose base location is no longer dependent on the largest network nodes.

As noted above, timeliness is closely coupled with distance in determining the value and thus frequently the price of information. This has the greatest effect in time-critical markets, such as financial services, emergency planning, fast moving/urban transport systems and command and control systems. In the many information-centric applications where timeliness is less important the domination of distance has substantially diminished and been replaced by cost and accessibility considerations. There is no simple model that can represent such time-distance inter-dependency[1], but in broad terms we can agree to recognise explicitly the need for distance measures to include time, cost, and field data (flows) rather than just physical space.

Just as physical distance may provide a surrogate for the time and cost of physical transport of goods or people, so we may consider the notion of *Information Distance* as a surrogate for the time and cost of non-physical delivery of information and services to a given location in response to a specific request from that location. For the present we define *Information Distance*[2] as the lowest price obtainable in the market for the delivery of a pre-specified and unmodified (i.e. complete and error free) unit of information from a given source to a given destination (or set of destinations) in a given interval of time. If the delivery of a unit of information demands a positive acknowledgement and/or receipt of a unit of information in response, the sum of the delivery-plus-response price must be obtained.

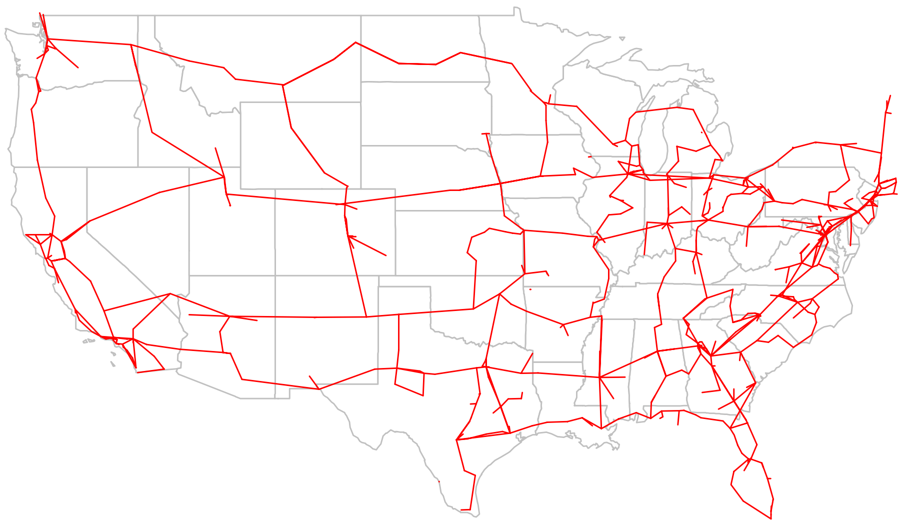
The unit of Information Distance is *price* (not cost, although the two may equate or be closely related). The parameters required for calculating Information Distance include the (logical and physical) location of the source, the (logical and physical) location of the destination, the requirements (technological, access) placed upon the destination by the source for satisfactory delivery, the specific information content sought (nature, volume) and the time parameters (start date/time, end date/time, delivery delay accepted or agreed, latency). From this it can be seen that a simple equation cannot be devised to provide a price, thus Information Distance is computed on a case-by-case basis, generally being derived from published tariffs and individual quotations. Such tariffs and quotations will vary according to the nature of the proposed purchasing decision – the length of contract, the size of contract, the location of the source and destination and the individual or organisation making the purchase. Therefore there is not strictly one Information Distance for a given user request, but many.

Information Distance (ID) clearly does not provide a metric or the basis for a metric space under most circumstances. In general, however, the definition does satisfy the co-location and separation rules, and in some cases may satisfy symmetry and triangularity rules. An example of a well-defined ID space would be the set of all public, active static IP addresses in the world, together with the price for the delivery to any one of these addresses of the latest live trade price and volume of all National Market shares on the Nasdaq exchange during market open hours with a delay of less than 2 seconds from their release by the Exchange's market data distribution systems – see further, Annex 4 - Traffic, teletraffic and statistical self-similarity.

Although the global availability of on-demand bandwidth[3] is far from perfect and in practice bandwidth and switching capacity is limited for many purposes, much traditionally location-bound information is now susceptible to distance-independent location – connectivity, access, permissioning, reliability, true throughput, time and cost dominate over distance. In fact *access* accounts for 35% of service provision in current US inter-city voice networking as compared with 15% for the long-haul networking and 50% is accounted for by areas such as marketing, billing and customer support. Thus 70% of the 'geographic' cost is accounted for by local access. Access is a far more dominant factor in less-developed regions of the world, especially in Africa, parts of Asia and South America. In many instances there is either no access at all or access at speeds and/or costs that preclude almost all users other than international corporations and government units.

This is not to say that network optimisation is no longer an issue in telecommunications – in fact it has been one of the most intensively researched areas of all over the last 25 years, with many key results being produced by the major telecommunications research labs, notably Bell Labs (now AT&T labs) in New Jersey[4]. It is to some extent the success of this work that is reflected in the continuing lowering of per bit costs and reduction of distance dependency, especially in IP data networks. Despite the fact that the Internet, ATM networks and private IP networks[5] generally provide distance-independent tariffs, underlying costs are not distance-independent – they rely heavily on optic fibre availability and links to key gateways which tends to focus high-intensity traffic into a small number of major cities (see for example, Figure 10-2).

Figure 10-2 AT&T's US inter-city optic fibre network links – 2001

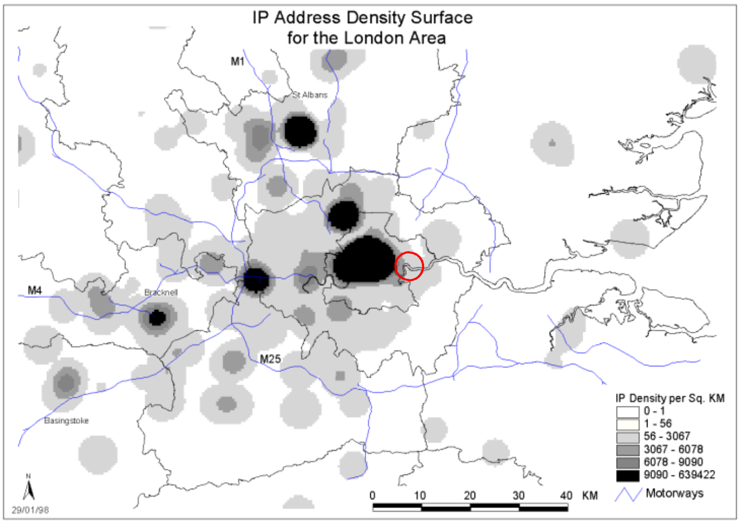


© AT&T

More traditional networks, such as leased line facilities, still exhibit strong distance dependency based on a fee for the two local tails/links to Central Office (CO) facilities plus a linear fee based on a straight line (Euclidean) distance between CO's. In many cases the distance component is tiered – relatively low cost, flat rate tariffs applying within close proximity to a suitable switch, coax cable or optic fibre route, and much higher rates (often a stepped or linear function of distance) for all other links.

The combined effect of these two factors results in 'clumping' of service availability, e.g. at secure telecentre facilities such as Telehouse in London's Docklands, similar to the patterns of joint demand and joint supply traditionally seen in retailing and manufacturing. Research by Martin Dodge and Narushige Shiode into the geography of ownership of Internet (IP) addresses in the United Kingdom[6] illustrates this visually for London (strictly speaking this provides a surrogate view of connectivity, Figure 10-3). The Docklands area is at the eastern edge of the main highest density zone shown in the centre of the map (see red circle highlight). Interestingly enough there is now a move eastwards from this zone for security reasons – the Docklands zone and the towers of Canary Wharf are within a security cordon and post September 11th 2001 are regarded as more vulnerable than nearby areas to the east. This is by no means the first example of security issues determining UK IT facility location, but it is one of the most vivid in recent years.

Figure 10-3 IP Address ownership density, London



© Martin Dodge, 1998-2002

It is important to note that these clumping effects are largely unaffected by the absolute cost per bit of information transmission, which has dropped substantially year on year for 25+ years. Demand, in the form of new services such as high resolution image transmission, streaming audio and video, voice over IP and streaming real-time data containing ever more detail (such as financial information and medical images) will more than make up for the per bit cost reductions. Overall, current statistics suggest that IP data traffic growth will dominate demand for at least the next 5 years and we are likely to see patterns such as those illustrated being re-enforced rather than dispersed over the coming years.

The release of dependence on distance for some activities, such as non real-time information retrieval, access to information services, inter-personal communication, ‘mail’, software development, graphic design, book editing etc. directly impacts the dynamics of the geographic space, rather as the introduction of motorways such as the M25 has impacted journey patterns. Figures released in June 2002 for UK employment show that approximately 2 million people or 7% of the working population now work at least one day a week from home, as telecommuters. In some industries, where people choose to live and work is less dependent on geography than access to good telecommunications infrastructures such as ISDN or ADSL. And vice versa, good telecommunications and IT infrastructures act as a pull factor for location decisions –

the concentration of high tech facilities in London's docklands is a recent example of this.

The telecommunications revolution diminishes or eliminates distances in a wide range of information-centric applications, including of course, the diffusion of innovations, but for many activities it has only an indirect effect (e.g. altering shopping patterns and delivery routes for certain classes of goods) or has very little short term effect at all. It can be argued, with today's ever-increasing congestion and pollution, that much remains to be done to resolve the so-called tyranny of space, and that computers and information technology are only beginning to have an effect, albeit minor. And in some respects, the impacts are negative, increasing congestion as they facilitate centralisation and provide information on ever more places to visit and work.

A fundamental challenge for Society is to devise ways of utilising new technologies to diminish the need for travel – for example, by providing direct electronic access to the *contents* of museums, libraries, galleries, shops and cinemas, not just access to their catalogues. Likewise, for the workplace, the development of real-time, large format audio-visual experiences and secure, economic remote access will enable practical teleworking and interaction with service providers to be extended to a large proportion of the populace, not just a minority for a subset of their time. This process needs to embrace public as well as private services, since in many countries access to such facilities is limited and very problematic, whilst in others they account for a large percentage of employment and economic activity. This applies especially to those services having direct contact with the public – medical services, education services and local and regional government.

It is to be hoped that developments such as these will lead, in time, to greater investment in homes and community facilities rather than regional and national transport infrastructures. This may not bring about the 'death of distance' but perhaps, a more desirable objective, the 'management of distance' based on environmental and social principles, and not merely economic and political pressures.

Notes and References

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[1] *see for example,* ***Murnion S*** *(**2000**) Cyberspatial analysis: appropriate methods and metrics for a new geography, Ch.13 in* ***Openshaw S, Abrahart R J*** *(**eds.**) (**2000**) Geocomputation, Taylor and Francis, London*

[2] *This definition is not related to the information theoretic use of the term, e.g. as per Kullback and others*

[3] ***Bandwidth****: the amount of information (typically in bits/second) that may be transmitted from point A to point B in a network over a given time period. Note that bandwidth values require specification of more than just a single number, since it is important to know what the end-to-end achieved throughput is, whether this applies to all times of day and durations of transmission or just subsets, whether the delivery and throughput is guaranteed or not etc.*

[4] *http://www.research.att.com/*

[5] ***ATM*** *– asynchronous transmission mode;* ***IP*** *– internet protocol*

[6] *http://www.cybergeography.org/*

11 Annex 1 - Nearest neighbour statistics

This Annex provides derivations for the probability distribution and moments of the n^{th}-nearest neighbour distance in *k*-dimensional space. The analysis is based on ideas initially drafted by Prof. M F Goodchild and the present author in 1974, the first part of which draws on the earlier work of Thompson[1] (1956), but which were not published at the time.

The nearest neighbour statistic is obtained by obtaining the average distance from a set of randomly selected sample locations to the closest observed data points (events) in the sample space (point to event distances). This problem is equivalent to the problem of finding the mean values where the events themselves rather than random points are used as the sample locations (event-event distances). If this average distance is calculated for the nearest point it provides the first-order nearest neighbour statistic; if it measures the average distance to the second or subsequent points it provides the second or n^{th}-order nearest neighbour statistic. If the observed point pattern is truly random throughout the sample space (commonly referred to as "Complete Spatial Randomness" or CSR) then the observed mean values will equal the expected, and their ratio will be unity.

The distributions and expected values, derived below, assume an unbounded space and a predefined (known) point density, λ. This assumption has two important drawbacks: the first is that sample spaces are always finite and boundary effects may be significant; the second is that selection of the sample space will influence the results. The latter occurs for two reasons: first, the sample space may (in fact will generally) exhibit variation which may result in non-random observed point distribution; and second, different partitions of the possible sample space may alter the result (i.e. similar to the well-known *modifiable areal unit problem*).

The basic solution to this problem for CSR has been known for many years[2], but has recently attracted renewed interest from mathematicians and physicists[3]. This is partly due to its wide range of applications: from the familiar areas of geography, ecology and epidemiology, to areas as varied as crime incident analysis[4], packing problems, cluster analysis, minimal spanning trees, stellar dynamics, archaeological research and the behaviour of liquids.

11.1.1 Linear case

Consider a line of infinite length, with points randomly distributed along it according to a Poisson process, $P(r)$, with mean density λ. Select one point, or location, at random. The probability that a point is found a distance $r+\partial r$ from this sample point may be given by:

$$P(r)\partial r = P(2r=0).P(2\partial r=1)$$

From the terms of the Poisson distribution we have:

$$P(r)\partial r = \frac{(2\lambda r)^0 e^{-2\lambda r}}{0!} \times \frac{(2\lambda \partial r)^1 e^{-2\lambda \partial r}}{1!}$$

thus, as ∂r tends to 0, we have

$$P(r)dr = \frac{e^{-2\lambda r}}{0!} \times \frac{(2\lambda dr)^1}{1!} = 2\lambda e^{-2\lambda r} dr \qquad A1.1$$

The mean of this distribution is given by:

$$\mu(r) = \int_0^\infty rP(r)dr = 1/2\lambda$$

and the variance is given by:

$$\sigma^2(r) = \int_0^\infty r^2 P(r)dr - \mu^2(r) = 1/4\lambda^2$$

By making the substitution, $2r\lambda = x/2$ in (1.1) above, the distribution can be seen to be a χ^2 distribution with 2 degrees of freedom:

$$P(x)dx = \frac{1}{2}e^{-x/2}dx \qquad A1.2$$

If we now sample *N* points and average the results, we obtain a better estimate of the mean nearest neighbour distance (reduced variance). This averaged value is simply:

$$\bar{r} = \frac{1}{N}\sum r = \frac{1}{4\lambda N}\sum x \qquad A1.3$$

The summation term in *x* is the sum of *N* χ^2 distributions with 2 degrees of freedom, which is a χ^2 distribution with *2N* degrees of freedom. The mean of this distribution is 2 and the variance, *2N*, so the mean and variances based on a sample of *N* are simply:

$$\mu(\bar{r}) = 1/2\lambda\,, \quad \sigma^2(\bar{r}) = 1/4\lambda^2 N \qquad A1.4$$

These results can be generalised to n^{th}-order nearest neighbours using the same method but in this case using the combined distribution:

$$P(r_n)\partial r_n = P(2r{=}n{-}1).P(2\partial r{=}1)$$

thus

$$P(r_n)\partial r_n = \frac{e^{-2\lambda r_n}(2\lambda r_n)^{n-1}}{n-1!} \times \frac{e^{-2\lambda\partial r_n}(2\lambda\partial r_n)^1}{1!} \qquad A1.5$$

and

$$P(r_n)dr_n = \frac{e^{-2\lambda r_n}(2\lambda)^n r_n^{\,n-1}}{n-1!}dr_n \qquad A1.6$$

In this case we find:

$$\mu(\bar{r}_n) = n/2\lambda\,, \quad \sigma^2(\bar{r}_n) = n/4\lambda^2 N \qquad A1.7$$

11.1.2 k-dimensional case

The linear case utilises the 'volume' of a *1*-dimensional hypersphere of radius, r, in its use of the term $2r$. The general expression for the volume of a k-dimensional hypersphere, V_k , is given by:

$$V_k = \frac{\pi^{k/2} r^k}{\Gamma(k/2+1)}$$

and thus incremental volume is:

$$dV_k = \frac{k\pi^{k/2} r^{k-1}}{\Gamma(k/2+1)} dr = \frac{kV_k dr}{r}$$

Using these expressions the k-dimensional problem can be solved in the same manner as the *1*-dimensional case:

$$P({}_k r_n)\partial r = P(V_k{=}n\text{-}1).P(\partial V_k{=}1)$$

from which we find:

$$P({}_k r_n)\partial r_n = \frac{e^{-\lambda V_k}(\lambda V_k)^{n-1}}{n-1!} \times \frac{e^{-\lambda \partial V_k}(\lambda \partial V_k)^1}{1!} \qquad A1.8$$

thus

$$P({}_k r_n) dr_n = \frac{\lambda e^{-\lambda V_k}(\lambda V_k)^{n-1}}{n-1!} dV_k \qquad A1.9$$

substituting for dV_k from above, we have

$$P({}_k r_n) dr_n = \frac{\lambda e^{-\lambda V_k}(\lambda V_k)^{n-1}}{n-1!} \frac{\lambda k V_k dr}{r} \qquad A1.10$$

Letting $\Phi r_n^{\,k} = V_k$ this becomes:

$$P({}_k r_n)dr_n = \frac{e^{-\lambda\Phi r_n{}^k}(\lambda\Phi r_n{}^k)^n k}{n-1!}\frac{dr_n}{r} \qquad A1.11$$

Thus, for first-order nearest neighbours in two dimensions, we have:

$$P(r) = 2\pi r\lambda e^{-\lambda\pi r^2}$$

The mean of the distribution in A1.11 can be found by integration:

$$\mu({}_k r_n)dr_n = \int_0^\infty \frac{e^{-\lambda\Phi r_n{}^k}(\lambda\Phi r_n{}^k)^n k}{n-1!} dr_n \qquad A1.12$$

By making the substitution, $x=\lambda\Phi r_n{}^k$ in (1.12) above, we have $dx=k\lambda\Phi r_n{}^{k-1}dr_n$ and thus:

$dx/ k\lambda\Phi r_n{}^{k-1}= dr_n$, giving $dr_n =rdx/x$, so that:

$$\mu({}_k r_n)dr_n = \int_0^\infty \frac{e^{-x}x^n}{n-1!}\frac{rdx}{x} \qquad A1.13$$

Now $r_n=(x/\lambda\Phi)^{1/k}$ and:

$$\mu({}_k r_n)dr_n = \frac{1}{n-1!(\lambda\Phi)^{1/k}}\int_0^\infty \frac{e^{-x}x^{n+\frac{1}{k}-1}}{n-1!}dx \qquad A1.14$$

so the mean value is given by:

$$\mu({}_k r_n)dr_n = \frac{1}{n-1!(\lambda\Phi)^{1/k}}\Gamma\left(\frac{nk+1}{k}\right) \qquad A1.15$$

which we can also write as:

$$\bar{r}_{n,k} = \frac{\Gamma(n+1/k)}{\Gamma(n)}\frac{1}{(\lambda V_k)^{1/k}} \qquad A1.16$$

and general crude moments, $\alpha = 1, 2, 3\ldots$ by:

$$\mu'({}_k r_n{}^{\alpha})dr_n = \frac{1}{n-1!(\lambda\Phi)^{\alpha/k}}\Gamma\left(\frac{nk+\alpha}{k}\right) \qquad A1.17$$

Notes and references:

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[1] ***Thompson H R (1956)*** *Distribution of distance to Nth neighbour in a population of randomly distributed individuals, Ecology, 37, 391-394*

[2] *it is believed to date back to Gustav Hertz (1887-1975) in 1909*

[3] ***Percus A G, Martin O C*** (***1998***) *Scaling universalities of k-th nearest neighbour distances on closed manifolds, Advances in Appl. Maths., 21, 424-436*

Evans D, Jones A J, Schmidt W M (***2002***) *Asymptotic moments of near neighbour distance distributions, Proc. Roy. Soc. Lond. A, 458, 1-11*

These recent papers confirm that the results we derive in this Annex provide the basic model for more general cases, such as where the set of points is a specific number, N, the sample space is a compact closed body ***C*** *rather than unbounded space, and the distribution of points in* ***C*** *may be selected according to any well-behaved sampling density. They also provide a model for the analysis of (hyperspherical) boundary effects by considering the additional case: P(k-1 sites exist at a distance≤ r) P(1 site exists at a distance= r+dr) and P(N-k sites exist at a distance > r+dr) within* ***C***

[4] *see for example,* ***Levine N (2002)*** *Crimestat II: A spatial statistics program for the analysis of crime incident locations, N Levine & Associates, Houston, TX and National Institute for Justice, Washington DC, http://www.icpsr.umich.edu/NACJD/crimestat.html (National Archive of Criminal Justice Data, USA)*

12 Annex 2 - Geodesics on cost surfaces

Let $s = \int_{\alpha}^{\beta} F ds = \int_{\alpha}^{\beta} F(x,y)\sqrt{g_{ij}dx^i dx^j}$ Equ. 12-1

where $dx^1 = dx$, $dx^2 = dy$ and the g_{ij} are coefficients of the summation (as described earlier in the text), and α and β are initial and final point on some curve or path, *C*, along which the integral is to be evaluated.

Now let $F = e^{\sigma}$ where $\sigma = \sigma(x,y) = log_e F$ then Equ. 1 becomes

$$s = \int_{\alpha}^{\beta} \sqrt{e^{2\sigma} g_{ij} dx^i dx^j}\, dt$$ where *t* is a parameter Equ. 12-2

Let $g^*_{ij} = e^{2\sigma} g_{ij}$ then the equation for incremental distance may be re-written as:

$$s = \int_{\alpha}^{\beta} \sqrt{g^*_{ij} dx^i dx^j}\, dt$$ Equ. 12-3

Minimising Equ.3 yields the geodesic equations[1] (parameterised in terms of *s*, arc length):

$$\frac{d^2x^l}{ds^2} + \left\{\frac{\overline{l}}{jk}\right\}\frac{dx_j}{ds}\frac{dx_k}{ds} = 0 \qquad (j, k, l = 1,2)$$ Equ. 12-4

where summation is again assumed over repeated indices and $\left\{\frac{\overline{l}}{jk}\right\}$ is the modified Christoffel symbol of the second kind[2]. This symbol can also be written as:

$$\left\{\frac{\overline{l}}{jk}\right\} = \left\{\frac{l}{jk}\right\} + \delta_j^l \frac{\partial\sigma}{\partial x^k} + \delta_k^l \frac{\partial\sigma}{\partial x^j} - g_{jk} g^{lm} \frac{\partial\sigma}{\partial x^m} \qquad \text{Equ. 12-5}$$

where $\delta_i^j = 0$ if $i \neq j$ and $\delta_i^j = 1$ if $i = j$, and the g^{ij} are the entries in the inverse matrix of g_{ij}*'s*. Substituting in Equ.4 and letting $G_l=0$ be the geodesic equations on the original surface we have:

$$G_l + \left(\delta_j^l \frac{\partial\sigma}{\partial x^k} + \delta_k^l \frac{\partial\sigma}{\partial x^j} \right) \frac{dx^j}{ds} \frac{dx^k}{ds} - g_{jk} g^{lm} \frac{dx^j}{ds} \frac{dx^k}{ds} = 0 \qquad \text{Equ. 12-6}$$

or

$$G_l + 2\frac{\partial\sigma}{\partial x^k} \frac{dx^l}{ds} \frac{dx^k}{ds} - g^{lm} e^{-2\sigma} \frac{d\sigma}{dx^m} = 0 \qquad \text{Equ. 12-7}$$

Multiplying Equ.7 *by* $(dy/dx)(ds/dx)^2$ when $l=1$ and by $(ds/dx)^2$ when $l=2$ and subtracting produces a single equation in terms of x and y alone[3]:

$$G + 2\frac{\partial\sigma}{\partial x^k} D\left(g^{lk} \frac{dy}{dx} - g^{2k} \right) = 0 \qquad \text{Equ. 12-8}$$

where $G=0$ is the geodesic of the original surface in terms of x and y, and

$$D = g_{11} + 2g_{12}y' + g_{22}(y')^2$$

Writing the second component of Equ.8 as R we have the desired result,

$$G + R = 0 \qquad \text{Equ. 12-9}$$

Now R may be written as

$$R = \frac{D}{Fg}\left(F_x\left(g_{22}y' + g_{12}\right) - F_y\left(g_{12}y' + g_{11}\right)\right)$$

where

$$g = \begin{vmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{vmatrix}$$

For the plane, $g_{11}=g_{22}=1$ and $g_{12}=g_{21}=0$ thus $g = 1$ and $D = 1 + (y')^2$ giving

$$R = \frac{1+(y')^2}{F}\left(F_x y' - F_y\right) \qquad \text{Equ. 12-10}$$

and $\quad G = y''$

12.1 Uniform and single variable cost functions

As noted above, the general equation $G+R=0$ is simplest for the case where the physical surface is a plane. In this case $G = y''$ and we have $y''+R=0$ as the set of equations to solve. Now the general form for R is:

$$R = (1+y'^2)(F_x y' - F_y)/F$$

where $F(x,y)$ is a generalised cost function defined over the sample region.

The single variable cost function in y is of the form:

$$z = F(x,y) = F(y)$$

Since z is a function of y only, partial derivatives with respect to x vanish, and Equ.10 gives:

$$R = -(1+y'^2)F_y / F$$

thus:

$$y'' = (1 + y'^2)F_y/F$$

We examine four cases in the sections that follow:

(i) F is a uniform (constant cost) function

(ii) *F* is a function of *y* or *x* alone

(iii) *F* is a simple function of *x* and *y* combined

(iv) The cost surface is re-mapped into polar coordinates (r,θ) and solutions for $F(r)$ are sought, i.e. radially symmetric cost surfaces or velocity fields. This latter case has been the subject of analysis and solution for particular cases by Angel and Hyman and by Wardrop (see further, below)

Solutions to piecewise linear models, such as a simple ridge, valley or marquee (roof-like) structure, can be derived from the results for continuous sub-regions. In such cases a series of *i* separate boundary value problems may need to be solved to find analytic solutions for the optimum route from *A* to *C* (*A* to B_i, and B_i to *C*).

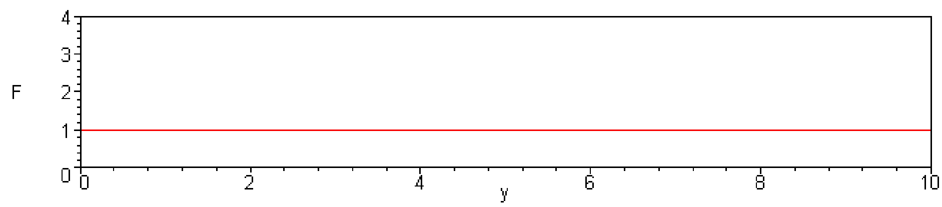
12.1.1 Uniform costs

If the cost function is uniform, i.e. F=a constant, then the differential equation to be solved is simply $G=0$. Shortest paths will therefore be straight lines. We can use this case as a base model for non-trivial cost functions, by taking a region of the plane and mapping paths about an initial point, say $(x_o,y_o) = (5,2)$, with a range of initial directions.

The directions we are using have the integer values over the range $y'(2)$ = [-10,10] i.e. a total of 21 directions. The solution equation is:

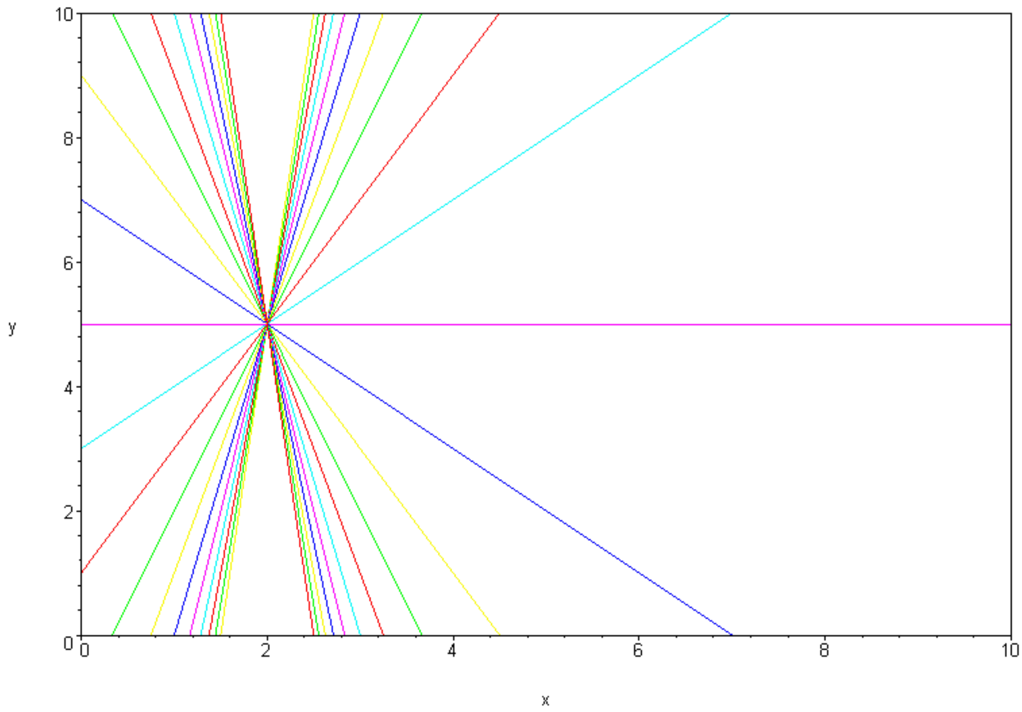
$$y(x) = 5 + y'(2).(x-2) \qquad \text{i.e.} \qquad y(x) = y_o + y'(2).(x-x_o)$$

The cost function, taking $F=1$ is shown below, for comparison with subsequent cost functions:



The path solution map over a sample space of y=[0,10], x=[0,10] is shown in Figure 12-1.

Figure 12-1 Geodesic paths for the plane with constant costs, at y(5)=2



12.1.2 Cost functions of one variable

If $F(x,y)$ is a function of y only, the expression for $G+R$, above, simplifies to:

$$y'' = (1 + y'^2)F_y / F$$

Let $w = 1 + y'^2$ then $w' = 2y'y''$ and substituting in the above equation we have:

$$w' = 2wy'F_y/F \text{ or}$$

$$w'/w = 2y'F_y/F$$

Integrating each side of the equation gives us $log_e w = log_e(kF)^2$ where k is a constant of integration and re-substituting for w, gives

$$log_e(1 + y'^2) = log_e(kF)^2$$

which can be re-written as:

$$\int \frac{dy}{\sqrt{k^2F^2-1}} = \int dx$$

For functions such as $F = ay + b$, $F = a/y$ and $F = ae^{py}$, $y(x)$ may be found by direct integration. These cases are examined in the following subsections. Higher powers in y, such as y^2, are not directly solvable (in the case of y^2 the solution is in the form of elliptic integrals). For more complex functions series approximation and/or numerical integration is required. If $F = 1/V$, for example where V is velocity, the above expression can be re-written as:

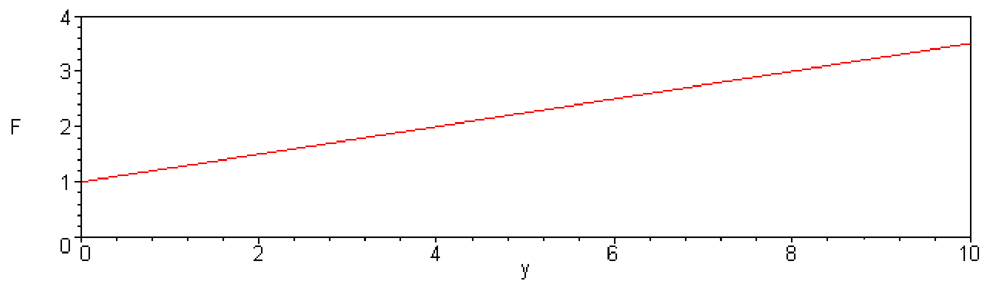
$$\int \frac{kVdy}{\sqrt{1-k^2V^2}} = \int dx$$

12.1.2.1 Linearly increasing costs – in one variable

The cost function, F, can take many forms, but the simplest non-trivial case is a linear increase in y, e.g. $F=ay+b$, $a>0$. The general solution of $G+R=0$ for this case is:

$$y(x) = \alpha \cosh(x/\alpha+\beta/\alpha)-b/a$$

where α and β are constants which are determined by the initial or boundary conditions imposed. For example, let $a=1/4$, $b=1$ then the cost function, F, is as shown below:



With these parameters the costs associated with any point of the sample space are at least 1 (when $y=0$), are constant for any given distance from $y=0$ (i.e. do not change with x), but increase steadily in the y-direction. As such they can be viewed as corresponding to increasing costs (or an environmental variable, such as noise or traffic congestion) in the neighbourhood of a linear feature, such as a highway, beachfront, urban region etc.

Inserting the selected values for *a* and *b* into the solution equation above we find:

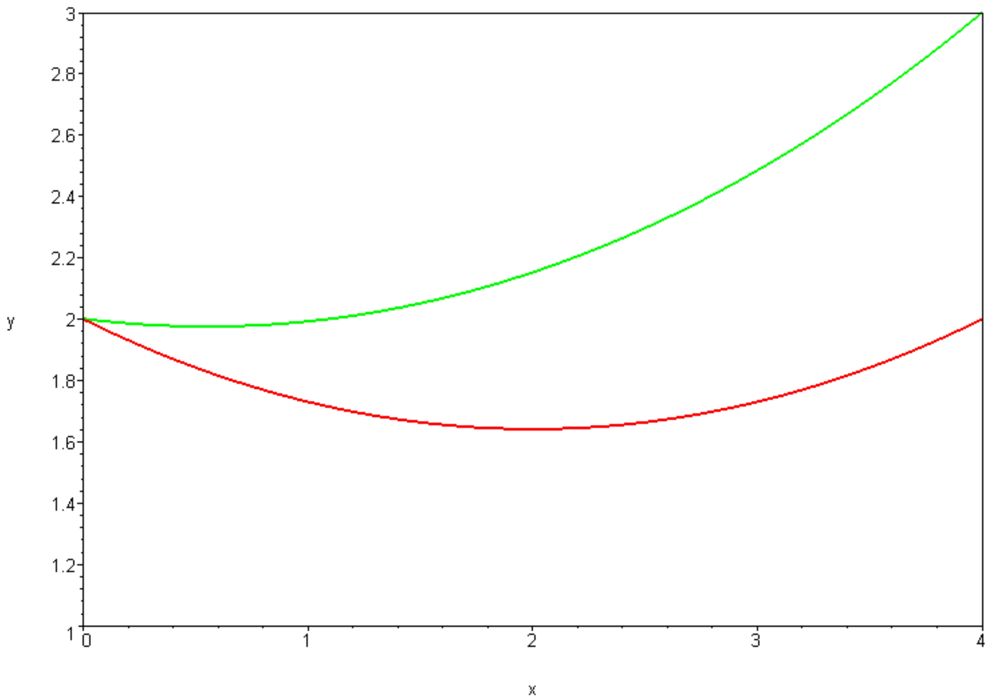
$y(x) = \alpha cosh(x/\alpha+\beta/\alpha)\text{-}4$

Suppose we wish to find the least cost path from (0,2) to (4,2). We find that this is satisfied by β=-2. Letting α=2/c and substituting for β we have:

$$y(x) = \frac{2}{c}\cosh\left((x-2)\frac{c}{2}\right) - 4$$

For $y(0)=2$ as an initial point, this yields $c=0.354498$ (determined by numeric methods). The resulting path is plotted below (see Figure 12-2), together with the path from (0,2) to (4,3) (In this case β=-0.55, $c=0.3347$). These paths are known as *catenary* or hyperbolic cosine curves. Note that if $b=0$ the final term in the path function disappears but otherwise is unaltered. Also note that the form of the solution applies for all values of $a>0$.

Figure 12-2 Least cost paths for *F=y/4+1*, from (0,2) to (4,2) and (0,2) to (4,3)



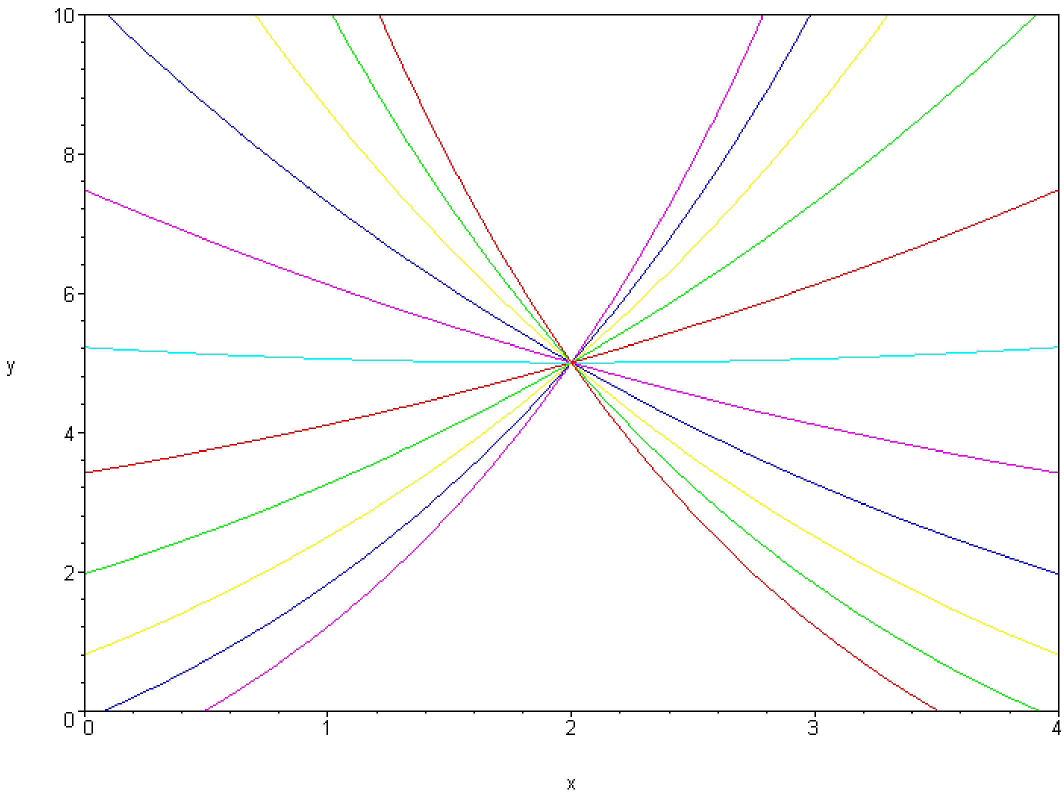
Path lengths and path costs in these two cases are clearly not equal. The straight-line (Euclidean) path *lengths* are 4.00 and √17 (c.4.12) respectively. The path *costs* along these straight-line paths are 6.00 and 6.70 respectively, whilst the path costs along the solution paths are 5.88 and 6.59, i.e. slightly lower cost despite the longer path. Path costs are computed by integrating *Fds* along the path selected (where $ds^2 = dx^2 + dy^2$).

Whilst closed form solutions are of inherent interest, it is not always easy to obtain the values for the constants by specifying boundary conditions (i.e. start and end points). A more general approach is to specify an initial condition (the location of the initial point) and then to calculate shortest path trajectories as a map based on varying the initial direction (first differential) systematically. This method, also known as a 'shooting method' for obvious reasons, can be applied to the analytic solution if a suitable form is available, or to a series approximation. Using a series solution of Taylor-Maclaurin type, and an initial point $y(2)=5$, we obtain a series in the above example of the form:

$$y(x) = 5 + y'(2)(x-2) + \frac{1}{18}(1+(y'(2))^2)(x-2)^2 + \frac{1}{486}(y'(2)+(y'(2))^3)(x-2)^3 + \ldots$$

i.e. a series in powers of $(x\text{-}2)$ with coefficients that are functions of the derivative at the starting point (this result is similar to that described earlier for the constant costs case). This series can then be evaluated for a range of $y'(2)$ values. In Figure 12-3 below the set of curves for integer values in the range $y'(2)$ =[$-5,5$] are shown based upon this series solution. The curved form of the paths clearly shows the divergence from those shown earlier in the linear case. Divergence from the constant costs model increases as the coefficient of y increases.

Figure 12-3 Geodesic paths for the plane with cost function *F=y/4+1*, at *y*(*5*)=*2*



Power series provide a very convenient solution approach but they suffer from lack of convergence at some distance from the initial point and/or with larger values of the first differentials at the initial point – in such cases a series approach is no longer acceptable.

Analytic solutions have the advantage that they apply across all non-singular values of the expression. Using this approach, solving for the initial point *y*(*2*)=*5* and then selecting a range of values for the second constant of integration, the map of least cost paths can be extended as far as required. This process is illustrated in the two figures below, for the linear case with *a=1/4* and *a=10* respectively. In the second example least cost paths between points that are not close are deflected very markedly towards the *x*-axis thereby taking advantage of the lower costs achievable, despite substantial increases in overall path length.

Figure 12-4 Geodesic paths for the plane with cost function *F=y/4+1*, at *y*(*5*)=*2*

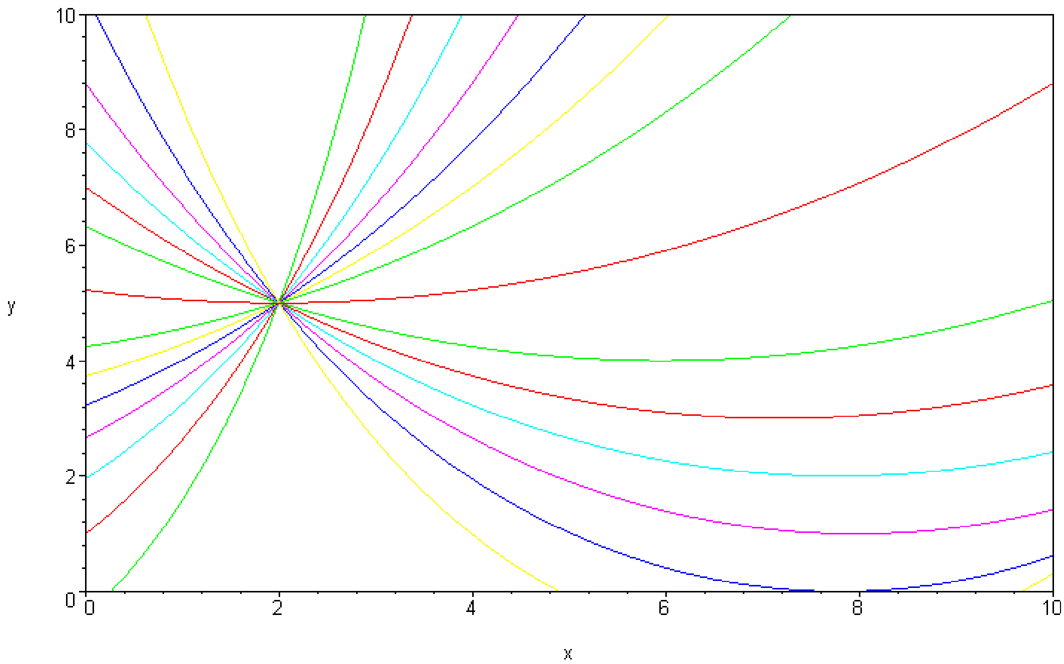
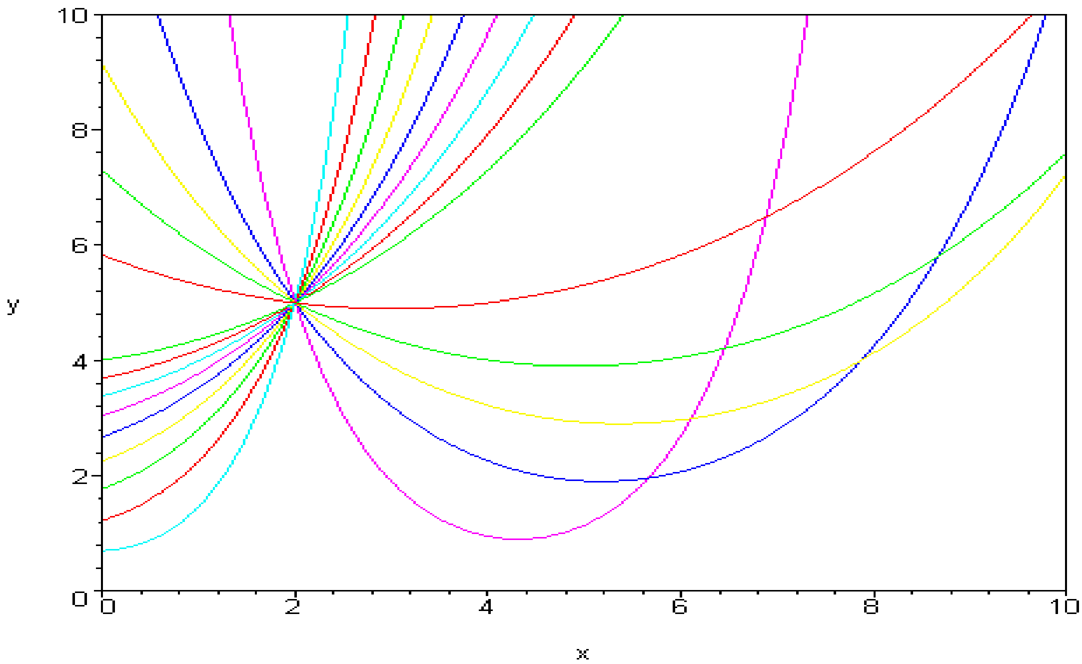
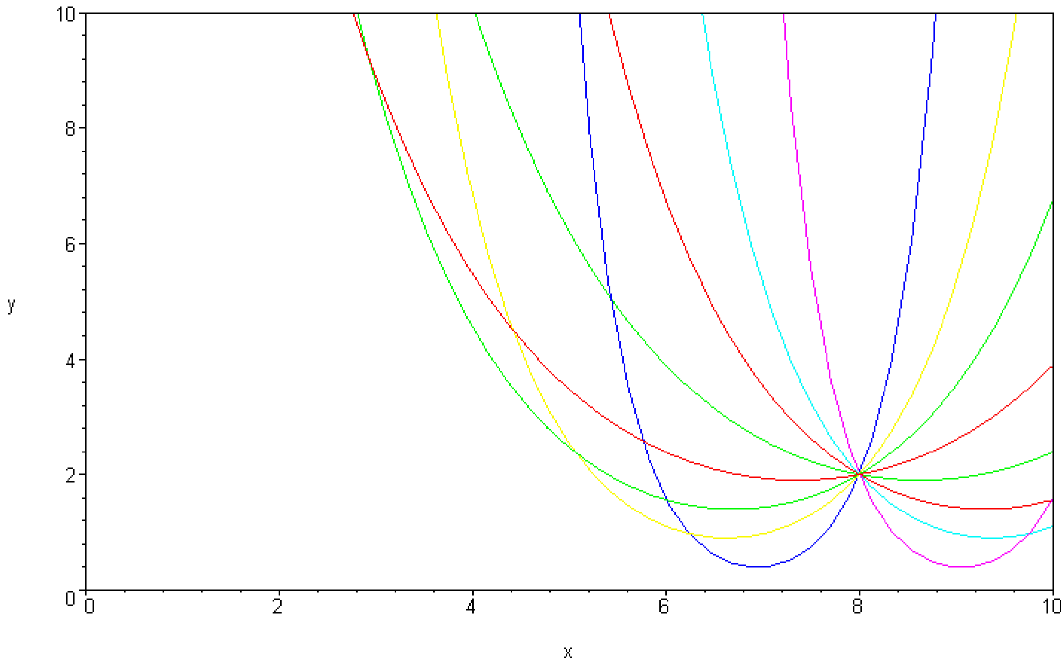


Figure 12-5 Geodesic paths for the plane with cost function *F=10y+1*, at *y*(*5*)=*2*



A number of observations can be made concerning the geodesic paths for *F=10y+1*. The first observation is that we use the term *geodesic paths* in this instance (and others below) even though the paths cross in some cases and thus may not satisfy the global Inf{} condition for geodesics we stated in Chapter 4 – each path shown is a geodesic in the neighbourhood of the initial point but may not be a geodesic over its entire length. The second observation is that when locations are reasonably close to one another, Euclidean straight-line paths provide a good approximation to optimal routes. However, the excessive costs incurred when travelling by Euclidean paths to points that are more widely separated justify radical departures from direct routes. Indeed, there is a critical point at which the cost of a direct route equals that of a totally different path, as can be seen above by considering paths from (2,5) to (7,6). In travel terms, if we equate this function to lower congestion where y is small (further from the core of the town or city) this equates to the question "is it worth driving across town to my destination or shall I go out to one of the ring routes and back in along an arterial route"? There are clearly quite large regions of the solution space where uncertainty predominates, i.e. a number of possible routes which may be radically different will have similar costs or travel times. A further issue relates to reachability. In the model illustrated, there is a region of the solution space in the lower right around (8,2) which does not appear to be reachable from (2,5). Solving from (8,2) confirms this observation (see below):

Figure 12-6 Geodesic paths for the plane with cost function *F=10y+1*, at *y*(*8*)=*2*

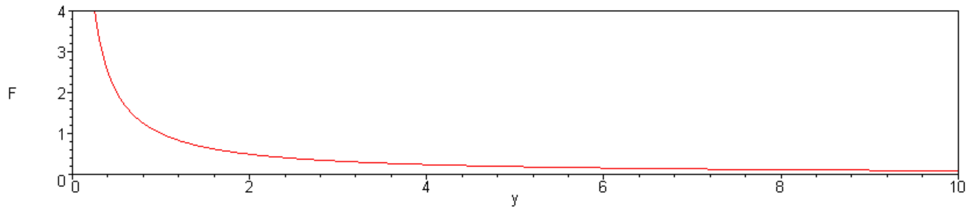


Clearly, optimal solution paths do exist but they involve paths that extend beyond the limits of the solution region into areas where $y<0$, or more correctly, a piecewise solution comprised of an arc from (2,5) to (x_1,0), then from (x_1,0) to (x_2,0) and then from (x_2,0) to (8,2), where $2 < x_1 < x_2 < 8$.

If the cost function varies with x alone, the solution paths will be of inverse hyperbolic cosine form. Goodchild[4] (1977) analysed this case and gave a solution in logarithmic form.

12.1.2.2 Inverse cost function – one variable

If $F=a/(y+b)$ the cost function is at a maximum when $y=0$ (where it equals a/b, $b>0$) and diminishes rapidly with increasing y, as shown below:



This means that paths, which bend away from the x-axis, will be less costly, being the reverse of the previous example. The solution to this problem is available in closed form and does not involve the constant, a. Evaluating the result as before with an initial value of $y(2)=5$ gives:

$$y(x) = -b \pm \sqrt{b^2+10b+29 + 4\alpha - 2\alpha x - x^2}$$

where α is a constant of integration, as before. Setting $b=0$ and squaring the equation to remove the square root shows that the solution is of the form:

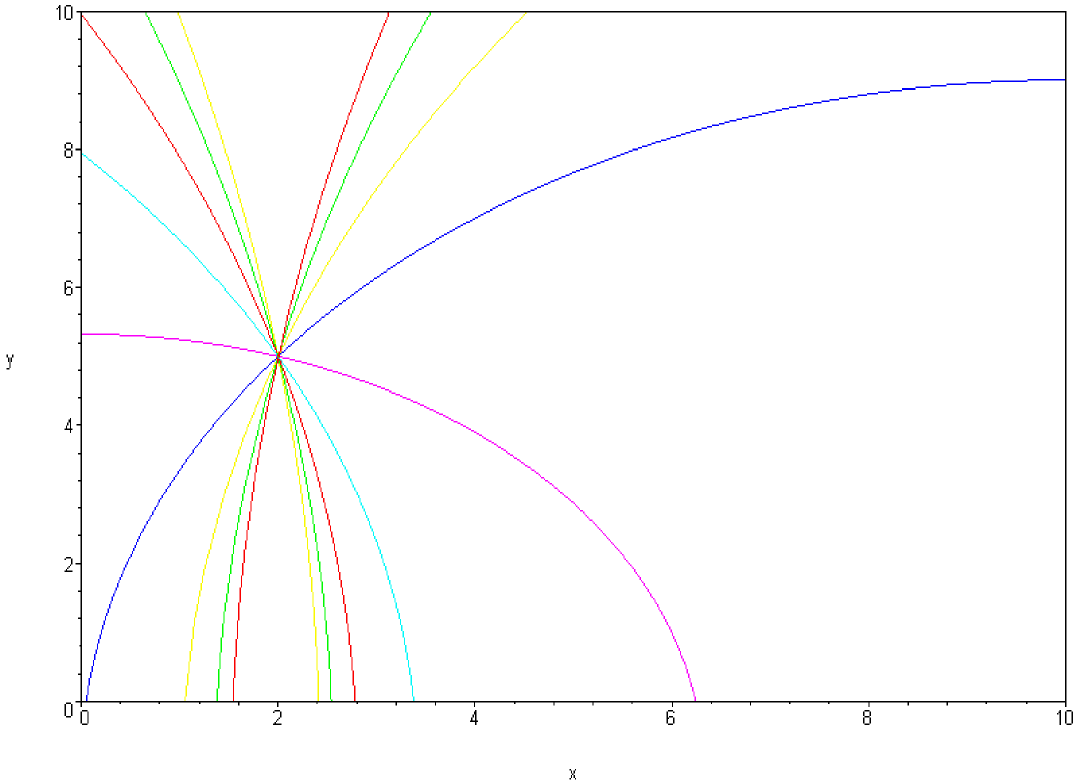
$$(x-2)^2 + y^2 = r^2 \text{ where } r^2=29 \text{ in this case}$$

i.e. solution paths are arcs of circles or straight lines (when the y-values are equal). In fact, this result is well known in non-Euclidean geometry, since the expression

$$ds = \frac{1}{y}\sqrt{dx^2 + dy^2}$$

is the metric of hyperbolic or Lobachevsky space. Furthermore, if $F=1/y$ represents variation in velocity over the region, this result shows that shortest *time* paths are arcs of circles with curvature towards regions of higher travel speeds. This result is illustrated below, using the same initial point and range as before (the arcs appear stretched but this is simply a result of differences in the x- and y-scales):

Figure 12-7 Geodesic paths for the plane, cost function *F=a/(y+1)*, at *y(2)=5*



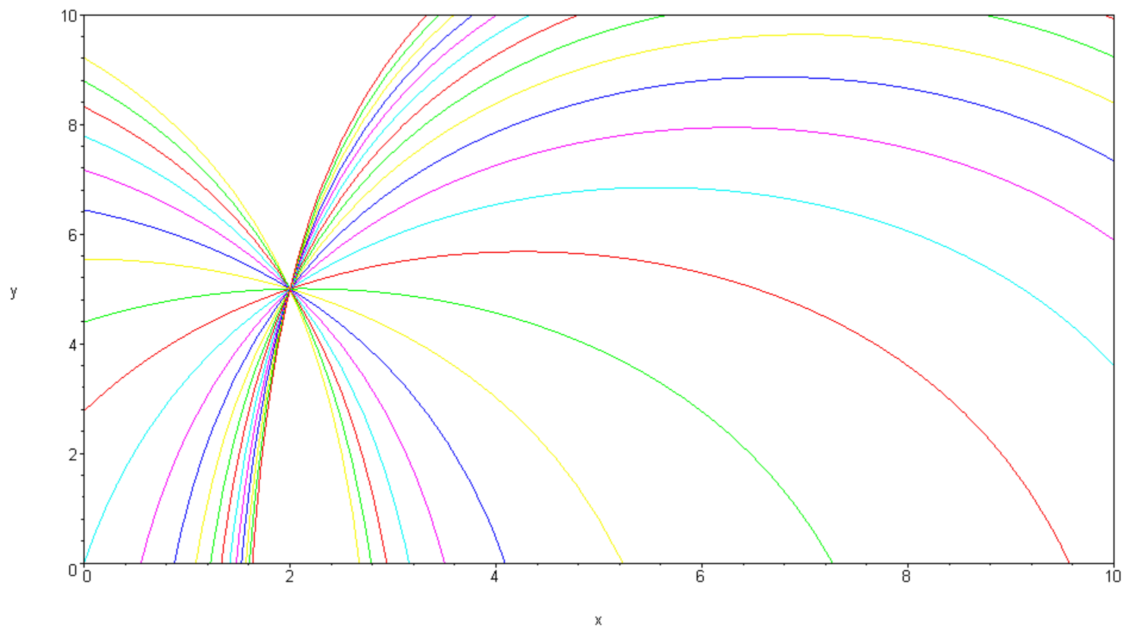
12.1.2.3 Exponential function – one variable

If $F(x,y) = ae^{py}$ then a family of cost functions is defined which vary with the choice of p. If p is negative costs decrease exponentially as y increases, and vice versa for p positive. The solution curves are of the form:

$$y(x) = -\frac{1}{p}\ln(\alpha\sin(px) - \beta\cos(px))$$ where α and β are constants as before

Setting $y(2)=5$ and $p=-0.25$ we can generate a family of solution paths as shown below.

Figure 12-8 Geodesic paths in cost field $F=ae^{-y/4}$ at $y(2)=5$



There is a clear similarity between this set of geodesic paths and the previous one, where solution paths were circle arcs. Optimal paths are diverted towards the lower cost region (higher values of y) in smooth arcs, with the degree of divergence from straight routes increasing with the separation of the initial and target locations.

12.2 Bivariate cost functions

A general bivariate linear cost function over the plane has the form $F=ay+bx+c$. This function is a tilted plane over the underlying uniform physical surface. A closed solution to this problem does not appear achievable. A series solution of the type described earlier can be found, but convergence is a problem.

Figure 12-9 below shows solution paths obtained using a series approach for the cost surface $F = y + x$ in the neighbourhood of $x=2$, for values of $y'(2)=[-2,2]$. The solution path for $y'(2)=1$ is $y(x) = 3 + x$, i.e. a straight line, since the rate of cost change is constant in this direction. For $y'(2)=2$, however, the polynomial solution is:

$$y(x) = 1 + 2x - \frac{5(x-2)^2}{14} + \frac{10(x-2)^3}{49} - \frac{1175(x-2)^4}{8232} + \frac{535(x-2)^5}{4802}$$

which does not converge for values of x which are much greater than 2. This finding confirms the earlier observation that in many cases series solutions of this form can only be used in the neighbourhood of the initial point.

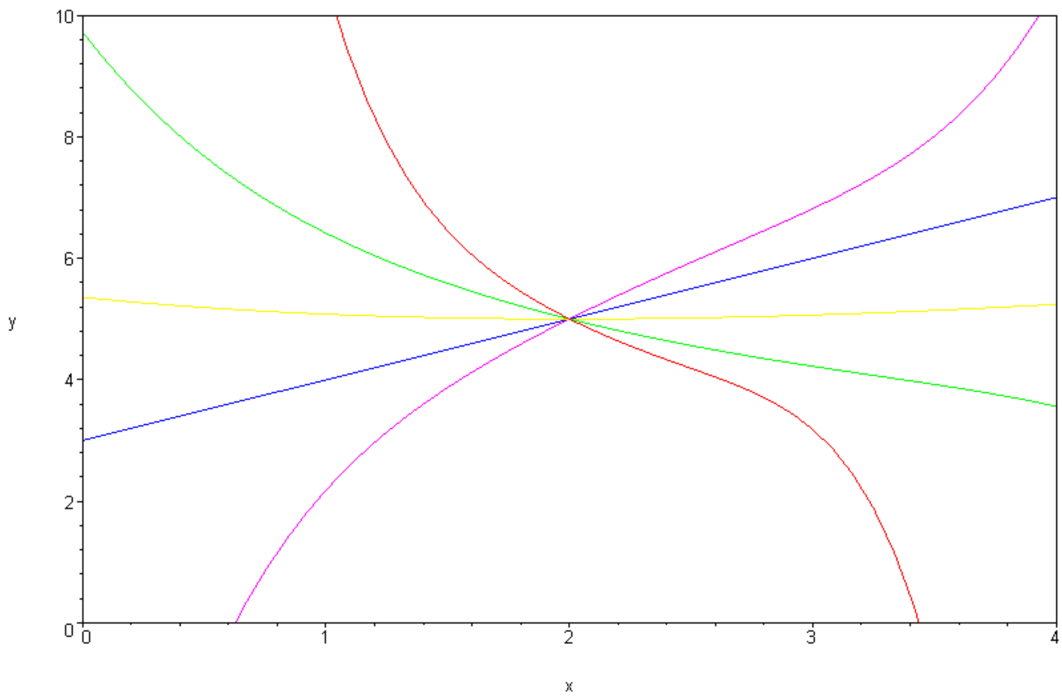
In this example, inspection of the solution paths shows:

(i) there is always one solution path through a given point that is a Euclidean straight line (with direction ∇F)

(ii) paths are asymmetric in form around a given initial point, as one would expect, and

(iii) path shapes retain the hyperbolic cosine (cosh and $\cosh^{-1}$) curve structures of the solutions found for costs varying in y or x alone, but have a more pronounced curvature.

Figure 12-9 Geodesic paths for the plane with cost function *F=y+x*, at *y*(*5*)*=2*



12.3 Radially symmetric cost functions

The radially symmetric cost function:

$$z = F(r, \theta) = F(r)$$

has been studied in detail by Angel and Hyman[5] and Wardrop[6]. In this case we find that the integral relation is of the form:

$$\int \frac{dr}{r\sqrt{k^2 r^2 F^2 - 1}} = \int d\theta$$

or, letting $F = 1/V$ where $V(r)$ is a radially symmetric velocity field, the relation is of the form:

$$\int \frac{KVdr}{r\sqrt{r^2 - K^2 V^2}} = \int d\theta$$

For the case $F = \omega r^p$ the solution path $r(\theta)$ may be found by direct methods. If $p=1$ the solution curves are derived by Angel and Hyman as log spirals, of the form $ln(r) = m\theta + c$ where m and c are constants determined by the two boundary values or an initial value and direction.

12.4 Composite cost functions

Cost functions may be constructed by combining several contiguous regions, each of which has a different cost function. If these discrete cost functions have no discontinuities at their edges then paths from one region to another will be smooth and not refracted at the boundaries - solution paths are determined by the point of crossing and the direction of travel at that point (i.e. an initial value problem). For example, a composite model of traffic in a city might consist of a central rectangular region of approximately constant speed (say 10mph), with a linear velocity field on either side (10mph increasing to, say, 55 mph at the city edge), and a similar radially symmetric linear velocity field at each end. This lozenge-shaped or *marquee* city model facilitates direct computation of least time paths from any point in the city to any other – these will be comprised of linear, circular and log spiral elements.

Notes and references:

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[1] ***Eisenhart L P*** (***1925***) *Riemannian Geometry, Princeton Univ. Press, Princeton, New Jersey, Section 17, "Geodesics", p.50 Equ. 17-8*

[2] ***Eisenhart L P*** (***1925***) *op. cit., Section 28 "Conformal spaces. Spaces conformal to a flat space"*

[3] *an alternative simplification could be achieved by reversing the operations in y and x*

[4] ***Goodchild M F*** (***1977***) *An evaluation of lattice solutions to the problem of corridor location, Environment and Planning A, 9, p.732*

[5] ***Angel S, Hyman G M*** (***1976***) *Urban Fields, Pion, London, pp.20-29*

[6] ***Wardrop J G*** (***1969***) *Minimum-cost paths in urban areas, reprinted in Angel and Hyman* (*op. cit.*), *pp.155-161*

13 Annex 3 – Sample algorithms

13.1 Distance transforms

13.1.1 Sample pseudo-code for 5x5 distance transforms

DT(,) is an array of lattice values, initialised to a large number, e.g. 9999 (greater than the maximum distance or cost distance) that will be generated by the algorithm; **DT(n,m)** = 0 is a set of 1 or more source points - e.g. a single point or set of points; xdim = number of rows–2; ydim = number of columns-2

LDM( ) is local distance matrix (mask entries), with values a1, a2 and a3 (integer or fractional), e.g. a1=2.2062, a2=1.414, a3=0.9866 or a1=11, a2=7, a3=5

DX( ) is an array of row movements and **DY( )** an array of column movements, identifying the position in the mask to select

Define masks, then scan DT array adjusting the distances until least local distance is selected; note that forward scan starts from row+2 col+2 and similarly for backwards scan, to allow for mask size

In these examples, **p** is a scaling factor representing the lattice size (e.g. for a 25m DEM it would be 25), and
DEM(,) is a topographical adjustment array (e.g. DEM values), i.e. if p=1 and DEM(,)=constant then d1 = d +LDM(k) and the transform is a standard DT.

COST(,) is an array of generalised costs or velocity field values, with all entries >0

```
Forward scan:
Data: LDM(0) to LDM(8) = [a1;a1;a1;a2;a3;a2;a1;a3;0]
Data: DX(0) to DX(8) = [-2; -2; -1; -1; -1; -1; -1; 0; 0]
Data: DY(0) to DY(8) = [-1;1;-2;-1;0;1;2;-1;0]

increment i = 2 to xdim
increment j = 2 to ydim
  d0=DT(i,j)
  increment k = 0 to 8
    r=i+DX(k)
    c=j+DY(k)
    d=DT(r,c)
       if standard DT use
              d0=min(d+LDM(k),d0)
       else use one of the alternatives below:

       if variable topography problem include following lines:
              s = LDM(k)*p
              t = DEM(r,c)-DEM(i,j)
              s1 = s*s
              t1 = t*t
              s2 = square-root(s1+t1)
              d1 = d+s2
              d0=min(d1,d0)
       end of inclusion

       if variable cost/velocity field problem include following lines:
              d0=min(d+LDM(k)*(COST(r,c)),d0)
       end of inclusion
```

```
   next k
          if problem includes obstacles, code obstacles as XXXX (greater than max distance) and include
          the following lines:
                    if DT(i,j)<XXXX then
                              DT(i,j)=d0
                    end if
          else use
                    DT(i,j)=d0
          end if
next j
next i

Backwards scan:
Data: LDM(0) to LDM(8) = [0;a3;a1;a2;a3;a2;a1;a1;a1]
Data: DX(0) to DX(8) = [0; 0; 1; 1; 1; 1; 1; 2; 2]
Data: DY(0) to DY(8) = [0;1;-2;-1;0;1;2;-1;1]

increment i = xdim to 2 step -1
increment j = ydim to 2 step -1
   d0=DT(i,j)
   increment k = 0 to 8
      r=i+DX(k)
      c=j+DY(k)
      d=DT(r,c)
          if standard DT use
                    d0=min(d+LDM(k),d0)
          else use one of the alternatives below:

          if variable topography problem include following lines:
                    s = LDM(k)*p
                    t = DEM(r,c)-DEM(i,j)
                    s1 = s*s
                    t1 = t*t
                    s2 = square-root(s1+t1)
                    d1 = d+s2
                    d0=min(d1,d0)
          end of inclusion

          if variable cost/velocity field problem include following lines:
                    d0=min(d+LDM(k)*(COST(r,c)),d0)
          end of inclusion
   next k
          if problem includes obstacles, code obstacles as XXXX (greater than max distance) and include
          the following lines:
                    if DT(i,j)<XXXX then
                              DT(i,j)=d0
                    end if
          else use
                    DT(i,j)=d0
          end if
next j
next i
```

for non-standard DT cases, iterate above until change in all DT(i,j) = 0 or < preset small value; to track solution paths record the values of DX(k) and DY(k) which are selected as the min values, storing the results in new arrays x(i,j) and y(i,j) or in an additional dimension of DT(i,j)

13.1.2 Sample MATLAB code for exact distance transform

```
% Exact Euclidean Distance Transform
% Uses MATLAB image processing toolbox function bwdist() with default  transform type (Euclidean)
% Variation: Multiple weighted Distance Transform example - Decision diagram problem
% Note: arrays of 1000x1000 or larger may be used, with sustained performance (seconds)
home;tic;

% define initial binary image with two object points (stations) and weights for stations (w1) and school (w2)
bw = zeros(101,101); w1=1;w2=1.5; bw(90,11) = 1;  bw(70,81) = 1;

% generate exact distance transform using built-in image toolbox function, first image
D = bwdist(bw);

% generate second distance transform
bw = zeros(101,101);  bw(30,31) = 1;  E = bwdist(bw);

% generate weighted combination of transforms
G = w1*D + w2*E;

% report solution minimum value
p = min(min(G))
[i,j]=find(G==p)

% report elapsed processing time (computational phase)
toc

% shaded contour plot
tic ; whitebg([0 .4 .6]); rect = [100, 100, 650, 600];
figure(1); set(1,'Position',rect); str = datestr(now,0);
contourf (G,15); colormap jet; alpha (.3); title([' Generated on: ',str]); xlabel('x-position'); ylabel ('y-position');
hold on;
plot(j,i,'-ys','LineWidth',4,'MarkerEdgeColor','k','MarkerFaceColor','w','MarkerSize',10);
plot(11,91,'--rs','LineWidth',2,'MarkerEdgeColor','k','MarkerFaceColor','k','MarkerSize',10);
hold on;
plot(81,70,'--rs','LineWidth',2,'MarkerEdgeColor','k','MarkerFaceColor','k','MarkerSize',10);
hold on;
plot(31,30,'--rs','LineWidth',2,'MarkerEdgeColor','k','MarkerFaceColor','k','MarkerSize',10);

% report elapsed processing time (graphical rendering phase)
toc
```

13.2 RRT and Vortal

This section contains sample code extracts (C++) for RRT and the VORTAL Steiner optimisation problem described in the text. The procedure involves defining a triangle with or without rectangular obstacles, and then seeking an optimal path connecting all three vertices of the triangle by an intermediate point within the bounding triangle or at a vertex, avoiding the obstacles by a predefined proximity variable. the test program (steiner.exe) used in this example can be found at www.desmith.com/MJdS/index.html

A random tree is generated from each vertex in turn. The function, GetNextRRTPoint(), is repeated for each vertex until it is possible to join two of the three paths (they are within 'proximity' distance of one another). The third path is then connected to the nearest vertex of the first two connected paths, i.e. this step is treated as a two point RRT problem.

Once the three RRT paths have been determined and are connected by an intermediate point, an iterative path optimisation process is invoked as follows:

(i) the DoOptimizeTight() function simulates 'pulling' on each path in turn, removing unnecessary intermediate points if the path length (cost) is reduced by so doing;
(ii) the DoOptimizeSteiner() function then moves the connection point of the three 'tightened' paths to see if the overall distance/cost can be reduced; and
(iii) the function DoOptimizeSteinerArrayMain() moves each intermediate point around until it achieves no further improvement. A variant of this operation (not shown in this extract) checks to see if intermediate points can be moved to 'corners' of the solution space (corners of obstacles) – this can result in further improvement of the optimisation

Steps (ii) and (iii) are iterated until no further improvement is achieved.

The core of this first part of the algorithm is the generation of new points:

GetNextRRTPoint()

```
CVertex* CSteinerWnd::GetNextRRTPoint( const CVertex* pVertexCurrent, const CVertex* pVertex1, const
CVertex* pVertex2, const CPtrArray& array )
{
        CPoint    pntRandom;
        CPoint    pntTemp;
        double dAlpha                   = 0.0;
        double dCurrentDistance = GetDistance( pVertexCurrent->GetX(), pVertexCurrent->GetY(), pVertex1-
        >GetX(),
                pVertex1->GetY(), pVertex2->GetX(), pVertex2->GetY() );
        double dNewDistance             = 0.0;
        CVertex* pNearestVertex         = NULL
        int         i                   = 0;
        int         nNearestIndex       = 0;

        do {
                // get a random point in the state space
                pntRandom.x = GetNextX();
                pntRandom.y = GetNextY();
```

```
				// the new point must be at least one step size away from the old one
				if ( GetDistance( pVertexCurrent->GetX(), pVertexCurrent->GetY(), pntRandom.x,
					pntRandom.y ) >= m_nWalkStep )
				{
					// find nearest point in previously generated vertices
					nNearestIndex	= FindNearest( pntRandom, array );
					pNearestVertex	= (CVertex*)array.GetAt( nNearestIndex );
					pntTemp.x		= pNearestVertex->GetX();
					pntTemp.y		= pNearestVertex->GetY();

					if (pntRandom.x == pNearestVertex->GetX())
					{// point is in the same horizontal line
						pntTemp.y = pNearestVertex->GetY() + m_nWalkStep;
					}
					else if (pntRandom.y == pNearestVertex->GetY())
					{// new point is in the same vertical line
						pntTemp.x = pNearestVertex->GetX() + m_nWalkStep;
					}
					else
					{// get angle between current point and the random point
						dAlpha = GetAngle( pntTemp, pntRandom );

					// use the angle to calculate next point at m_nWalkStep distance
						GetPointFromAngle( pntTemp, pntRandom, dAlpha, m_nWalkStep );
					}
					//check if the point is in our triangle
					if ( PtInTriangle( pntTemp ) )
					{// Point is in triangle - continue with the calculation.
					 // re-calculate current distance as a distance between the
					 // nearest point and the destination points
						dCurrentDistance = GetDistance( pNearestVertex->GetX(), pNearestVertex-
							>GetY(), pVertex1->GetX(), pVertex1->GetY(), pVertex2->GetX(),
								pVertex2->GetY() );

						// now, when we have the next point - check if the distance
						// between CURRENT point and the DESTINATION points is less then
						// distance between NEXT point and the DESTINATION points
						dNewDistance = GetDistance( pntTemp.x, pntTemp.y, pVertex1->GetX(),
							pVertex1->GetY(), pVertex2->GetX(), pVertex2->GetY() );
						if (dCurrentDistance > dNewDistance)
						{// improvement in distance
						 // check if new point is too close to any of the obstacles
							if (IsTooCloseToObstacles( pntTemp ))
							{
								dNewDistance = dCurrentDistance + 1.0;
								continue;
							}
						}
					}
					i++;
				}
			} while(i<MAX_ITER_NUM && dCurrentDistance <= dNewDistance );
			// repeat calculation until NEW distance is less then the OLD distance

			if ( i >= MAX_ITER_NUM || dNewDistance == 0 )
			{// this point is wrong. The algorithm tried too many times to find next point
				return NULL;
			}
			else
			{
				ASSERT( pntTemp.x >=0 && pntTemp.y >= 0);
				CVertex* pVertex = new CVertex( pntTemp.x, pntTemp.y, pNearestVertex );
				return pVertex; // return new distance between two points
			}
}
//returns random X coordinate within the given boundaries
int CSteinerWnd::GetNextX() const
{
			int nX;
			do {
				nX = m_pRand->IRandom(m_nLeftBoundary, m_nRightBoundary);
			} while( nX < m_nLeftBoundary || nX > m_nRightBoundary);
			return nX;
```

```
}
// returns random Y coordinate within the given boundaries
int CSteinerWnd::GetNextY() const
{
	int nY;
	int i = 0;
	do {
		nY = m_pRand->IRandom(m_nTopBoundary, m_nBottomBoundary);
	} while( nY < m_nTopBoundary || nY > m_nBottomBoundary);
	return nY;
}
// returns angle in radians between two points
double CSteinerWnd::GetAngle( const CPoint& pnt1, const CPoint& pnt2 ) const
{
	return atan2(  fabs(pnt1.y - pnt2.y),  fabs(pnt2.x - pnt1.x) );
}
// use the angle to calculate next point at m_nWalkStep distance
void CSteinerWnd::GetPointFromAngle( CPoint& pntCurrent, const CPoint& pntRandom, double dAlpha, int
nStep ) const
{
	if (pntRandom.x > pntCurrent.x)
	{
		pntCurrent.x += cos( dAlpha ) * nStep;
	}
	else
	{
		pntCurrent.x -= cos( dAlpha ) * nStep;
	}
	if (pntRandom.y > pntCurrent.y)
	{
		pntCurrent.y += sin( dAlpha ) * nStep;
	}
	else
	{
		pntCurrent.y -= sin( dAlpha ) * nStep;
	}
}
```

The next step is to 'tighten' the random tree paths by removing vertices:

DoOptimizeTight()

```
/*
	Tight rope optimisation. Each path (array) must be optimised.
*/
double CSteinerWnd::DoOptimizeTight( )
{
	double dDist = OptimizeTightPartial( m_ResultingArray1 );
	dDist += OptimizeTightPartial( m_ResultingArray2 );
	dDist += OptimizeTightPartial( m_ResultingArray3 );
	return dDist;
}
/*
	Tight-line optimisation applied only on one path (array)
*/
double CSteinerWnd::OptimizeTightPartial( CPtrArray& arrResulting )
{
	// get current distance from starting point to ending point
	double dOldDistance = GetDistance( arrResulting );
	CPtrArray arrTmp;
	int i, nStart = 0;
	CVertex* pVertex1 = NULL;
	CVertex* pVertex2 = NULL;
	CVertex* pVertexNew = NULL;
	CVertex* pParent = NULL;
	BOOL bGetNew = TRUE;

	while( nStart < arrResulting.GetSize())
	{
		if (bGetNew)
		{
			pVertex1 = (CVertex*)arrResulting.GetAt( nStart );
			pVertexNew = new CVertex( pVertex1->GetX(), pVertex1->GetY(), pParent );
```

```
			arrTmp.Add( pVertexNew );

			pParent = pVertexNew;
		}
		else
		{
			bGetNew = TRUE;
		}

		nStart ++;
		// check each subsequent point if a valid line can be
		// drawn from pVertex1 to pVertex2
		for( i=nStart; i<arrResulting.GetSize(); i++)
		{
			pVertex2 = (CVertex*)arrResulting.GetAt( i );
			//if the new line is not a valid one. Get one step back, and
			//draw the line between the starting vertex and the current one
			if (!IsLineOK( pVertex1->GetX(), pVertex1->GetY(), pVertex2->GetX(), pVertex2->GetY()
))
			{
				if (nStart == i)
				{
					pVertexNew = new CVertex( pVertex1->GetX(), pVertex2->GetY(),
						pParent );
					arrTmp.Add( pVertexNew );
					pParent = pVertexNew;

					pVertexNew = new CVertex( pVertex2->GetX(), pVertex2->GetY(),
						pParent );
					arrTmp.Add( pVertexNew );
					pParent = pVertexNew;
				}
				else
				{
					CVertex* pVertexPrevious = (CVertex*)arrResulting.GetAt( i-1 );
					if ( IsLineOK( pVertex1->GetX(), pVertex1->GetY(), pVertexPrevious-
						>GetX(), pVertex2->GetY() ) )
					{
						pVertexNew = new CVertex( pVertexPrevious->GetX(),
							pVertex2->GetY(), pParent );
						arrTmp.Add( pVertexNew );
						pParent = pVertexNew;
						nStart = i - 1;
						bGetNew = FALSE;
						pVertex1->SetX( pVertexNew->GetX() );
						pVertex1->SetY( pVertexNew->GetY() );
						break;
					}
					else if ( IsLineOK( pVertex1->GetX(), pVertex1->GetY(), pVertex2-
							>GetX(), pVertexPrevious->GetY() ) )
					{
						pVertexNew = new CVertex( pVertex2->GetX(),
							pVertexPrevious->GetY(), pParent );
						arrTmp.Add( pVertexNew );
						pParent = pVertexNew;
						nStart = i - 1;
						bGetNew = FALSE;
						pVertex1->SetX( pVertexNew->GetX() );
						pVertex1->SetY( pVertexNew->GetY() );
						break;
					}
					else
					{
						nStart = i - 1;
						break;
					}
				}
			}
		}
		if ( i == arrResulting.GetSize() )
		{
			pVertex1 = (CVertex*)arrResulting.GetAt( i-1 );
			pVertexNew = new CVertex( pVertex1->GetX(), pVertex1->GetY(), pParent );
			arrTmp.Add( pVertexNew );
			nStart = i;
```

```
                }
        }
        double dNewDistance = GetDistance( arrTmp );
        // if new distance is less then the previous one the optimisation is OK
        if ( dNewDistance < dOldDistance )
        {// BINGO! new distance is better
                for( i=0; i<arrResulting.GetSize(); i++)
                {
                        pVertex1 =  (CVertex*)arrResulting.GetAt( i );
                        delete pVertex1;
                }
                arrResulting.RemoveAll();
                for( i=0; i<arrTmp.GetSize(); i++)
                {
                        pVertex1 =  (CVertex*)arrTmp.GetAt( i );
                        arrResulting.Add( pVertex1 );
                }
                arrTmp.RemoveAll();
                Invalidate();
                dOldDistance = dNewDistance;
        }
        else
        {// nothing! we have to delete the temp array.
                for( i=0; i<arrTmp.GetSize(); i++)
                {
                        pVertex1 =  (CVertex*)arrTmp.GetAt( i );
                        delete pVertex1;
                }
                arrTmp.RemoveAll();
        }
        return dOldDistance;
}
```

We now have a valid set of three paths, with partially optimised length, but which can be improved further by selecting alternative locations for the intermediate point at which the paths are joined and by moving the vertices that comprises each path. These two optimisations are computed iteratively until no further improvement in the total path length/cost if found.

Initially we move the intermediate point:

DoOptimizeSteiner ()

```
/*
        Moves connection point until the best solution is achieved
*/
double CSteinerWnd::DoOptimizeSteiner()
{
        if (m_ResultingArray1.GetSize() > 1 &&
                m_ResultingArray2.GetSize() > 1 &&
                m_ResultingArray3.GetSize() > 1)
        {
                CVertex*  pVertex1  = (CVertex*)m_ResultingArray1.GetAt( m_ResultingArray1.GetSize() - 2 );
                CVertex*  pVertex2  = (CVertex*)m_ResultingArray2.GetAt( m_ResultingArray2.GetSize() - 2 );
                CVertex*  pVertex3  = (CVertex*)m_ResultingArray3.GetAt( m_ResultingArray3.GetSize() - 2 );

                double dOldDistance =
                        GetDistance( pVertex1->GetX(), pVertex1->GetY(), m_pConnectionVertex->GetX(),
                                m_pConnectionVertex->GetY() ) +
                        GetDistance( pVertex2->GetX(), pVertex2->GetY(), m_pConnectionVertex->GetX(),
                                m_pConnectionVertex->GetY() ) +
                        GetDistance( pVertex3->GetX(), pVertex3->GetY(), m_pConnectionVertex->GetX(),
                                m_pConnectionVertex->GetY() ) ;

                double    dDist                  =       dOldDistance+1.0;
                int                  i           =       0;
                BOOL      bBetterSolutionFound = FALSE;

                while ( dDist != dOldDistance && ++i<1000)
                {
                        dOldDistance = dDist;
```

```
                    dDist = OptimizeSteinerPartial( dOldDistance, pVertex1, pVertex2, pVertex3 );
                    if (dDist < dOldDistance)
                    {
                              bBetterSolutionFound = TRUE;
                              UpdateWindow();
                    }
          }
          if (bBetterSolutionFound)
          {
                    //update connection points
                    pVertex1  = (CVertex*)m_ResultingArray1.GetAt( m_ResultingArray1.GetSize() - 1 );
                    pVertex2  = (CVertex*)m_ResultingArray2.GetAt( m_ResultingArray2.GetSize() - 1 );
                    pVertex3  = (CVertex*)m_ResultingArray3.GetAt( m_ResultingArray3.GetSize() - 1 );

                    pVertex1->SetX( m_pConnectionVertex->GetX() );
                    pVertex1->SetY( m_pConnectionVertex->GetY() );
                    pVertex2->SetX( m_pConnectionVertex->GetX() );
                    pVertex2->SetY( m_pConnectionVertex->GetY() );
                    pVertex3->SetX( m_pConnectionVertex->GetX() );
                    pVertex3->SetY( m_pConnectionVertex->GetY() );
                    Invalidate();
          }
          return    GetDistance( m_ResultingArray1 ) +    GetDistance( m_ResultingArray2 ) +
                              GetDistance( m_ResultingArray3 );
     }
     return 0;
}
```

OptimizeSteinerPartial()

```
#define   OPTIMISATION_STEP                   1
```

double CSteinerWnd::OptimizeSteinerPartial(double dOldDistance, CVertex* pVertex1, CVertex* pVertex2, CVertex*
pVertex3)

```
{
          double dDist = dOldDistance;
          dDist = SteinerPartial( -OPTIMISATION_STEP, -OPTIMISATION_STEP, dDist, pVertex1, pVertex2, pVertex3
);
          dDist = SteinerPartial( 0, -OPTIMISATION_STEP, dDist, pVertex1, pVertex2, pVertex3 );
          dDist = SteinerPartial( OPTIMISATION_STEP, -OPTIMISATION_STEP, dDist, pVertex1, pVertex2, pVertex3
);
          dDist = SteinerPartial( -OPTIMISATION_STEP, 0, dDist, pVertex1, pVertex2, pVertex3 );
          dDist = SteinerPartial( OPTIMISATION_STEP, 0, dDist, pVertex1, pVertex2, pVertex3 );
          dDist = SteinerPartial( -OPTIMISATION_STEP, OPTIMISATION_STEP, dDist, pVertex1, pVertex2, pVertex3
);
          dDist = SteinerPartial( 0, OPTIMISATION_STEP, dDist, pVertex1, pVertex2, pVertex3 );
          dDist = SteinerPartial( OPTIMISATION_STEP, OPTIMISATION_STEP, dDist, pVertex1, pVertex2, pVertex3
);
          return dDist;
}
```

SteinerPartial()

double CSteinerWnd::SteinerPartial(int nDeltaX, int nDeltaY, double dOldDistance, CVertex* pVertex1, CVertex*
pVertex2, CVertex* pVertex3)

```
{
          double dTemp = dOldDistance;
          dTemp =  GetDistance( pVertex1->GetX(), pVertex1->GetY(), m_pConnectionVertex->GetX()+nDeltaX,
                              m_pConnectionVertex->GetY()+nDeltaY ) +
                    GetDistance( pVertex2->GetX(), pVertex2->GetY(), m_pConnectionVertex->GetX()+nDeltaX,
                              m_pConnectionVertex->GetY()+nDeltaY ) +
                    GetDistance( pVertex3->GetX(), pVertex3->GetY(), m_pConnectionVertex->GetX()+nDeltaX,
                              m_pConnectionVertex->GetY()+nDeltaY ) ;
          if ( dTemp < dOldDistance )
          {
                    //check if the corrected lines are all OK
                    if (IsLineOK(  pVertex1->GetX(), pVertex1->GetY(), m_pConnectionVertex->GetX()+nDeltaX,
                                        m_pConnectionVertex->GetY()+nDeltaY )
                              && IsLineOK(  pVertex2->GetX(), pVertex2->GetY(), m_pConnectionVertex-
>GetX()+nDeltaX,
                                        m_pConnectionVertex->GetY()+nDeltaY )
                              && IsLineOK(  pVertex3->GetX(), pVertex3->GetY(), m_pConnectionVertex-
>GetX()+nDeltaX,
```

```
                        m_pConnectionVertex->GetY()+nDeltaY ) )
        {
                m_pConnectionVertex->SetX( m_pConnectionVertex->GetX() + nDeltaX );
                m_pConnectionVertex->SetY( m_pConnectionVertex->GetY() + nDeltaY );
                Invalidate();
                return dTemp;
        }
    }
    return dOldDistance;
}
```

We now move the points that comprise each of the three paths. In this case the points are maintained in an array.

DoOptimizeSteinerArrayMain()

```
double CSteinerWnd::DoOptimizeSteinerArrayMain()
{
    double dDist = DoOptimizeSteinerArray( m_ResultingArray1 );
    dDist += DoOptimizeSteinerArray( m_ResultingArray2 );
    dDist += DoOptimizeSteinerArray( m_ResultingArray3 );
    return dDist;
}
```

DoOptimizeSteinerArray()

```
double CSteinerWnd::DoOptimizeSteinerArray( CPtrArray& array )
{
    double dOldDistance = GetDistance( array );
    double dDistance    = dOldDistance;
    int                 k = 0;
    //CVertex*          pVertex = NULL;
    BOOL    bBetterSolutionFound = TRUE;
    BOOL    bFound1 = FALSE;
    BOOL    bFound2 = FALSE;

    if ( array.GetSize() <= 2)
    {
        return dOldDistance;
    }

    int j = 0;
    while ( j++ < 10 && bBetterSolutionFound)
    {
        bBetterSolutionFound = FALSE;
        for( int i=array.GetSize()-2; i>0; i-- )
        {
            k = 0;
            bFound1 = TRUE;
            while ( bFound1 && ++k<1000)
            {
                bFound1 = FALSE;
                dOldDistance = dDistance;
                dDistance = OptimizeSteinerArray( dOldDistance, i, array );

                if (dDistance < dOldDistance)
                {
                    bBetterSolutionFound = TRUE;
                    bFound1 = TRUE;
                    UpdateWindow();
                }
                else
                {
                    bFound1 = FALSE;
                }
            }
        }

        for( i=1; i<array.GetSize()-1; i++ )
        {
            k = 0;
            bFound2 = TRUE;
            while ( bFound2 && ++k<1000)
```

```
					{
						bFound2 = FALSE;
						dOldDistance = dDistance;
						dDistance = OptimizeSteinerArray( dOldDistance, i, array );

						if (dDistance < dOldDistance)
						{
							bBetterSolutionFound = TRUE;
							bFound2 = TRUE;
							UpdateWindow();
						}
						else
						{
							bFound2 = FALSE;
						}
					}
				}
			}
			return GetDistance( array );
}
```

Move the selected point to each of 8 neighbouring locations, based on the predefined optimisation step size and local 3x3 neighbourhood:

OptimizeSteinerArray()

```
double CSteinerWnd::OptimizeSteinerArray( double dOldDistance, int nIndex, CPtrArray& array )
{
	double dDist = dOldDistance;
	dDist = SteinerPartialArray( -OPTIMISATION_STEP, -OPTIMISATION_STEP, dDist, nIndex, array);
	if (dDist < dOldDistance) dOldDistance = dDist;
	dDist = SteinerPartialArray( 0, -OPTIMISATION_STEP, dDist, nIndex, array);
	if (dDist < dOldDistance) dOldDistance = dDist;
	dDist = SteinerPartialArray( OPTIMISATION_STEP, -OPTIMISATION_STEP, dDist, nIndex, array);
	if (dDist < dOldDistance) dOldDistance = dDist;
	dDist = SteinerPartialArray( -OPTIMISATION_STEP, 0, dDist, nIndex, array);
	if (dDist < dOldDistance) dOldDistance = dDist;
	dDist = SteinerPartialArray( OPTIMISATION_STEP, 0, dDist, nIndex, array);
	if (dDist < dOldDistance) dOldDistance = dDist;
	dDist = SteinerPartialArray( -OPTIMISATION_STEP, OPTIMISATION_STEP, dDist, nIndex, array);
	if (dDist < dOldDistance) dOldDistance = dDist;
	dDist = SteinerPartialArray( 0, OPTIMISATION_STEP, dDist, nIndex, array);
	if (dDist < dOldDistance) dOldDistance = dDist;
	dDist = SteinerPartialArray( OPTIMISATION_STEP, OPTIMISATION_STEP, dDist, nIndex, array);
	if (dDist < dOldDistance) dOldDistance = dDist;
	return dOldDistance;
}
```

14 Annex 4 - Traffic, teletraffic and statistical self-similarity

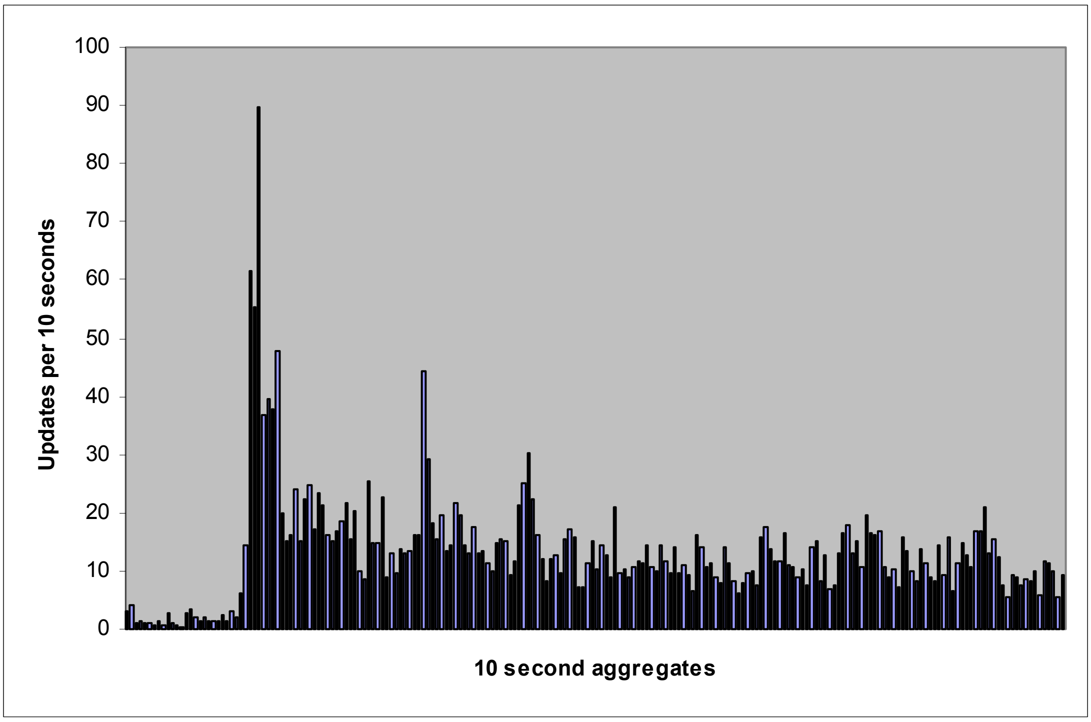
The statistical behaviour of telecommunications traffic (and LAN traffic) is the subject of an entire discipline known as teletraffic engineering and was one of the early foundation stones of statistics. Queuing theory, which originated in 19th century teletraffic analysis, provides models, solutions and insights for physical traffic. A core assumption of much of this analysis is that traffic arrival rates are independent and random (following a Poisson distribution). As with road traffic, teletraffic engineering is concerned with the rate of arrival of traffic into systems (switched networks) and the amount of time traffic spends within the system (hold times in session-oriented systems, transition times in session-less flows, and volume of data transmitted). Experience in teletraffic shows that a number of techniques can be implemented to manage flows more effectively – these include: ensuring the percent occupancy of facilities (e.g. links, switches and buffers) remains low, in many systems below 20% - this minimises the risk of collision, queuing and feedback effects such as retries; ensuring the system has multiple alternate routing options – this minimises the impact of link loss and provides diversity in case of unexpected peaks; smoothing design – this relates to a range of mechanisms, some content dependent, which aim to reduce the peak-to-average ratio, and includes the set of techniques known as traffic shaping. These techniques enable all (or more) traffic to be carried with greater reliability on a given capacity of network. In physical traffic engineering variable speed limit systems, traffic management schemes and multiple routes aim to have such effects. Feedback as a result of congestion may also have this effect but is less desirable as a short-term measure since it results in a degradation of service quality or throughput speed.

Recent research into the traffic characteristics of packetised data flows (e.g. IP traffic, LAN traffic, FTP file sizes etc) has shown that despite design efforts many types of data are far more peaky in their arrival rates and more long-tailed in their statistical distributions than one would expect with classical models[1]. The close relationship between long-tailed/heavy-tailed statistical distributions, power laws and long-range autocorrelation has suggested to some researchers that much data traffic is actually fractal in its nature.

Fractal (fractional dimension or more correctly, self-similar or self-affine) behaviour can be detected in a number of ways, notably by examining the long-range autocorrelation exhibited in the data[2]. As an example, we take the case of equity trade data output by the Nasdaq Stock Exchange. We can then consider the implications of such information for 'cybergeography' and traditional location theory.

The shares of Cisco are of particular interest, as they are amongst the most heavily traded on the Nasdaq market and also they are the principal supplier of the routing technology that underlies current IP networking. A sample dataset from 7th February 2002 was logged and the records from 13:00 – 15:00 GMT analysed. A total of just over 32000 records were transmitted, mostly trades (i.e. buys and sells) rather than price changes. Each record includes a date/timestamp to the nearest 1/100th of a second. These were analysed by calculating the frequency of updates, M, per 1, 10, 100 and 1000 second intervals (i.e. $\log M = 1,2,3,4$), and comparing this aggregation with the *log* of the variance at each level of aggregation (see Figure 14-1). The variance was consistently much larger than the mean (in a Poisson these would be the same) and the *logVar* decreased steadily for the whole sample from a high of 1.97 ($M=1$) to a low of 1.47 ($M=4$). This result is clearly heavily skewed towards the tail of the distribution, and shows a degree of self-similarity, but it would be wrong to ascribe this to any particular model at this stage.

Figure 14-1 Cisco (Nasdaq record updates)10 second aggregates, 14:27-15:00



Examining the data we see that there is an exceptionally large peak very shortly after 14:30:00 when the market opens, followed by a stabilisation after 2-3 minutes, with continued erratic behaviour thereafter but at a lower scale. The reason for this pattern is a reflection of the fact that almost all the data is generated by 'retail' order flow, that is, by private investors buying or selling relatively small numbers of shares (under a 1000 at a time in general). This trading process is conducted via their brokers to some extent, but to a larger degree by automated e-brokerage systems such as E-Trade, Schwab and similar online systems. Overnight, and before the Nasdaq market opens, buyers and sellers are placing orders with a resultant rush at market open when execution commences. It reflects a growing *dis-intermediation* in the marketplace, with regional brokerages being replaced in their order management role by centralised automated systems, connected to a single centralised electronic market. In a sense this is an aspect of globalisation and reflects simultaneously a trend towards increased *centralisation* (of certain core services) coupled with increased *decentralisation* (of demand). Modern telecommunications infrastructures and associated computing facilities thus have a complex role in re-shaping the geography of supply, demand and service provision, acting in at least three ways simultaneously, i.e.: *dis-intermediation, centralisation* and *decentralisation*.

Whilst there is definite evidence for unifractal and possibly multi-fractal behaviour of data communications traffic the jury remains out on this question[3]. What it does highlight is that fractal analysis techniques which originated in mathematical, financial and spatial research, turn out to have application in the statistical modelling of telecommunications, and almost certainly in physical traffic flows as well – cyberspace and cybergeography has unexpected links with traditional spatial models and both can provide insights and models for each other.

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[2] *The finite* (*bounded*) *nature of exchange data and much teletraffic limits the range of correlation considered. For trading data limited hours operation* (*e.g. 08:00 – 16:30 in London*) *remains the norm.*

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16 Internet links

16.1.1.1 Principal links, by subject (alphabetic)

AT&T/Bell labs research:
http://www.research.att.com/
Computational geometry:
Geometry in action:
http://www.ics.uci.edu/~eppstein/geom.html
Voronoi diagrams:
http://okabe.t.u-tokyo.ac.jp/okabelab/Voronoi/index.html
CASA
http://www.casa.ucl.ac.uk/about/index.htm
http://www.casabook.com/
GPS and earth rotation:
UK and European GPS:
http://www.gps.gov.uk
Galileo:
http://europa.eu.int/comm/dgs/energy_transport/galileo/index_en.htm
GPS and altitude:
http://mtp/jpl.nasa.gov/notes/altitude/altitude.html
International Earth Rotation Service (IERS):
http://hpiers.obspm.fr/
Graph theory and network problems:
http://www.nada.kth.se/~viggo/wwwcompendium/wwwcompendium.html
Lighthouses:
http://www.iala-aism.org/mainsite/index.html
http://www.lighthousemuseum.co.uk
www.nlb.org.uk
LOLA home page:
http://www.mathematik.uni-kl.de/~lola/
Magnetic variation:
http://Earth.agu.org/sci_soc/campbell.html
http://www.ngdc.noaa.gov/IAGA/wg8/igrfhw.html
Maps of cybergeography:
http://www.cybergeography.org/
Mathematical biographies:
http://www-gap.dcs.st-and.ac.uk/~history/
MATLAB:
http://www.mathworks.com
National Library of Scotland, "Charting the Nation" (Scotland) and Scapa Flow maps:
http://www.nls.uk/maps
http://www.chartingthenation.lib.ed.ac.uk/mapscot.html
Scapa Flow:
http://www.undiscoveredscotland.co.uk/eastmainland/churchill/
Robotics:
Automated animated characters:
http://robotics.stanford.edu/~kuffner/anim/index.html
"RRT":
http://msl.cs.uiuc.edu/rrt/
Spatial statistics tools including point pattern analysis:
Splancs (Diggle et al)
http://www.maths.lancs.ac.uk/Software/Splancs/
http://www.csiss.org/clearinghouse/select-tools.php3

Crimestat II (Levine):
http://www.icpsr.umich.edu/NACJD/crimestat.html

UCSB/NCGIA-sited projects:

Varenius:
http://www.ncgia.ucsb.edu/varenius/varenius.html

Vital:
www.ncgia.ucsb.edu/vital

UK Hydrographic office:
http://www.ukho.gov.uk/index.html

UK Road statistics:
http://www.transtat.dtlr.gov.uk/roadtraf/

Vincenty algorithms and terrestrial distances:
http://www.auslig.gov.au/geodesy/datums/calcs.htm
http://www.census.gov/cgi-bin/geo/gisfaq?Q5.1

16.1.1.2 Links for selected authors

Prof G Borgefors (Uppsala University, Sweden):
http://www.cb.uu.se/~gunilla/publications.html

Dr O Cuisenaire (École Polytechnique Fédérale de Lausanne)
http://ltswww.epfl.ch/~cuisenai/DT/

Prof M F Goodchild (UCSB, USA):
http://www.geog.ucsb.edu/~good/

Dr D S Johnson (AT&T Labs, USA):
http://www.research.att.com/~dsj/pub.html

Dr R Kimmel (Technion – Israeli Institute of Technology, Haifa):
http://www.cs.technion.ac.il/~ron/pub.html

Dr J S B Mitchell (SUNY, Stony Brook, USA):
http://www.ams.sunysb.edu/~jsbm/publications.html

Prof N Rowe (US Naval Postgraduate School, Monterey, California):
http://www.cs.nps.navy.mil/people/faculty/rowe/index.html